Single-Crossing Average Contracts and the Direction of Risk-Taking

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Abstract

This paper studies how contracts affect risk-taking under expected utility. We identify a class of contracts – single-crossing average (SCA) contracts – for which a simple local condition on the average slope determines the direction of risk-taking for all risk-averse agents. Contracts that amplify losses more than gains discourage risk-taking, while those that attenuate losses more than gains encourage it. The result does not rely on global concavity of the contract and accommodates general nonlinear transformations. SCA contracts are closed under composition, so their effects on risk-taking are preserved under layering. We also propose a local riskiness index that provides a complete ordering of contracts. This contract index is closely related to Yaari (1969)'s acceptance frontier, and admits a natural interpretation alongside the Arrow–Pratt measure of risk aversion. (JEL: D81, D86, D01, G11)

Keywords: risk-taking, contracts, risk aversion, single-crossing average.

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1 Introduction

How do contracts affect risk-taking? By reshaping the distribution of outcomes, contracts can alter the incentives of agents facing uncertainty. In many economic environments—such as executive compensation, asset management, and insurance design—contracts transform gains and losses asymmetrically, thereby influencing agents' willingness to undertake risky actions. Understanding when such transformations discourage or encourage risk-taking is a central question in economics and finance.

This paper studies this question in a general expected utility framework. We consider an agent with initial wealth w who evaluates a risky prospect X through a strictly increasing utility function. A contract is a mapping f that transforms realizations of X into $f(X)$. We ask: when does a contract make an agent more or less willing to take risks?

The main result shows that a local condition on the average slope $\frac{f(x)}{x}$ fully determines the direction of risk-taking for all risk-averse agents. We identify a broad class of contracts—single-crossing average (SCA) contracts—that characterize the direction of risk-taking for all risk-averse agents. The defining feature of this class is a local restriction on the average slope $\frac{f(x)}{x}$ around zero. Despite its local nature, this condition has strong global implications. Contracts that increase losses more than gains in an average sense discourage risk-taking at all wealth levels, while those that decrease losses more than gains encourage it. A simple asymmetry in the transformation of gains and losses determines the direction of risk-taking.

A key insight of the paper is that global properties of contracts are not required to obtain such results. In particular, the SCA condition imposes neither concavity nor monotonicity of the contract. Away from zero, SCA contracts can exhibit rich nonlinearities and discontinuities. The local behavior captured by the average slope around zero is sufficient to generate robust comparative statics. This sharply contrasts with existing approaches that rely on global curvature restrictions, such as concavification arguments (see, e.g., Ross (2004)).

To illustrate the distinction, we revisit a benchmark case: additive contracts of the form $f(x) = \alpha x$. When $\alpha > 1$, such contracts are known to discourage risk-taking for risk-averse agents, in the sense that agents under the contracts do not accept risks they would not accept in the absence of the contracts (see Gollier (2001); Hart (2011)). However, the additive contracts with $\alpha > 1$ need not "concavify" utility in the sense of making $u(f(x))$ a concave transformation of $u(x)$. This observation highlights a gap between transformations that alter curvature and those that systematically shift risk-taking incentives. The SCA class closes this gap by identifying the precise condition that governs the effect of contracts on risk-taking behavior.

In addition to this characterization, we establish several structural properties of SCA contracts. First, the class is closed under composition: if two contracts each discourage (or encourage) risk-taking, then their composition has the same effect. In contrast, we show that the closure property under addition holds only for contracts that discourage risk-taking, but not for those encouraging it. This property is particularly relevant in applications where multiple contractual layers interact. For example, modifying an existing contract can be viewed as a composite contract. A single firm may design and offer such contracts to its manager. On the other hand, when contracts are non-exclusive, multiple contracts that a single agent receives may be additive. Between the two situations, the overall "effective" contract may have different implications for agents' risk-taking.

Second, we show that the SCA condition is not only sufficient but also tightly connected to risk aversion. Under mild regularity conditions, if all SCA contracts discourage (or encourage) risk-taking, then the agent's utility must be concave. In this sense, SCA contracts provide a characterization of risk aversion in terms of comparative statics with respect to contractual transformations.

Finally, we propose a local riskiness index, defined in terms of the first and second derivatives of the contract at zero, which provides a complete ordering of contracts within a natural class. The proposed contract index is closely related to the concept of *the acceptance frontier* studied by Yaari (1969), and admits a natural interpretation of contracts alongside the Arrow–Pratt measure of risk aversion (Arrow (1974); Pratt (1964)). Using this index, we propose how to quantify and compare managerial compensation policies across firms.

Related literature. The paper relates to a literature that studies how transformations of risky outcomes affect risk-taking. Ross (2004) characterizes contracts that concavify utility, while Hart (2011) develops an approach based on acceptance and rejection criteria. While these approaches impose global restrictions on contracts or utility functions, our results show that a strictly weaker, local condition on contracts is sufficient to obtain global implications. More broadly, the paper connects to the theory of stochastic dominance (e.g., Levy (2016)), but differs in focusing on transformations of a given risk rather than general comparisons across risks.

Additive contracts play a central role in the principal-agent theory. Carroll (2015) establishes the optimality of additive contracts based on the robustness. In this paper, we do not formally study the optimality of contracts. Nevertheless, additive contracts arise as an important boundary case.[1] Because Carroll (2015) assumes risk-neutrality on both contracting parties, the issue of robustness *with respect to risk preference* is outside its scope. Hart (2011) studies this notion of robustness, and we follow his approach by employing wealth-uniform

[1] We discuss the contract optimality at the end of Section 4.

and utility-uniform concepts. Our work highlights the importance of additive contracts (and its generalization, SCA contracts) for risk-taking in the presence of diverse risk preferences.

A large finance literature studies how owners (principals) and managers (agents) share risky profits. That decision lies behind executive compensation, the firm's financing (e.g. debt-equity mix) and investment (e.g. takeover) strategies. The primary focus of this literature is the convexity of contracts (e.g. stock options) and how it affects various performance measures (e.g. share price variance). The evidence seems mixed.[2] In my view, there are two reasons behind this. First, as Ross (2004) emphasizes, the concave (convex) contracts do not generally make agents more (less) risk averse in the Arrow-Pratt sense. For the weaker notion of decreasing/increasing risk-taking studied in this paper, we show that the concavity/convexity of contracts is not even necessary. Second, the convexity/concavity is a global property of contracts, and as such it may be empirically hard to measure and detect. These considerations suggest that we should look for contract properties beyond the convexity/concavity that are relevant for risk-taking behaviors of a broad class of agents. We contribute to this literature by proposing a new *local* measure of contract riskiness that has a direct bearing on agents' risk-taking behavior.

The remainder of the paper is organized as follows. Section 2 introduces the framework and formal definitions. Section 3 studies additive contracts as a benchmark. Section 4 introduces SCA contracts and presents the main characterization. Section 5 develops the local riskiness index. Proofs not in the main text are collected in the Appendix A.

2 Contracts that Decrease or Increase Risk-Taking

We consider an agent with a strictly increasing utility function $u : \mathbb{R} \to \mathbb{R}$ and initial wealth $w \in \mathbb{R}$. The agent decides whether to undertake a risky prospect represented by a random variable X, which yields both positive (gains) and negative (losses) outcomes with strictly positive probabilities. Thus, final wealth $w + X$ may be above or below w.

A contract is a function $f : \mathbb{R} \to \mathbb{R}$ that modifies the realization of the risk. Under contract f, the agent evaluates the modified prospect $f(X)$ and bases the decision on expected utility $E[u(w + f(X))]$. We ask: under what conditions does a contract make the agent more or less willing to take risks relative to the absence of the contract?

To formalize this comparison, we introduce the following definitions.

Definition 1 *(Decreasing Risk-Taking) A contract f decreases risk-taking (**DRT**) for utility*

[2] For example, see Prendergast (1999, 2002), Guay (1999), Hemer et al. (2000) and other references in Ross (2004).

function u at wealth level w if

$$E\left[u\left(w+X\right)\right] \leq u\left(w\right) \;\; \Rightarrow \;\; E\left[u\left(w+f\left(X\right)\right)\right] \leq u\left(w\right) \tag{1}$$

for any random variable X such that both expectations are well defined.

Definition 2 *(Increasing Risk-Taking) A contract f increases risk-taking (**IRT**) for utility function u at wealth level w if*

$$E\left[u\left(w+X\right)\right] \geq u\left(w\right) \;\; \Rightarrow \;\; E\left[u\left(w+f\left(X\right)\right)\right] \geq u\left(w\right) \tag{2}$$

for any random variable X such that both expectations are well defined.

These definitions capture monotonic shifts in risk-taking behavior induced by contracts. A DRT contract makes the agent (weakly) more reluctant to accept any risk, while an IRT contract makes the agent (weakly) more willing to accept any risk.

From these definitions, it is immediate that degenerate risks impose basic restrictions on contracts. For example, suppose that X is a degenerate random variable that puts probability one at some non-positive outcome $x \leq 0$. Then the DRT condition (1) implies $f\left(x\right) \leq 0$ for that x. A simple sufficient condition for DRT is $f\left(x\right) \leq x$ for all x, which ensures that $f\left(X\right)$ is first-order stochastically dominated by X. Similarly, contracts satisfying $f\left(x\right) \geq x$ for all x are IRT for any increasing utility function.

Such contracts, however, are uninformative for our purposes, as they uniformly reduce or increase all outcomes. We therefore restrict attention to contracts that involve both positive and negative modifications of original outcomes.

Definition 3 *(Regular Contracts) A contract f is **regular** if there exist $x_1 < 0 < x_2$ such that $f\left(x_1\right) < 0 < f\left(x_2\right)$ and $\left(f\left(x_1\right) - x_1\right)\left(f\left(x_2\right) - x_2\right) < 0$.*

Regularity rules out two classes of degenerate contracts. The first condition excludes constant contracts by requiring variation in modified outcomes. The second condition imposes a weak trade-off between gains and losses: if the contract amplifies gains, it must also amplify losses; if it attenuates losses, it must also attenuate gains. This condition is sufficient to exclude contracts that induce first-order stochastic dominance.

Uniform Comparisons. For regular contracts to decrease or increase risk-taking, additional restrictions are required on either the utility function, the contract, or both. While such restrictions can be derived for a given pair (u, w), our goal is to identify conditions that apply more broadly across environments. To this end, we introduce stronger, uniform notions of DRT and IRT. We first eliminate dependence on the initial wealth level.

- *A contract f is **wealth-uniformly DRT for** u, if it is DRT for u at every w.*
- *A contract f is **wealth-uniformly IRT for** u, if it is IRT for u at every w.*

We then extend the requirement across classes of utility functions. Let U be a set of utility functions.

- *A contract f is U-**DRT**, if it is wealth-uniformly DRT for all $u \in U$.*
- *A contract f is U-**IRT**, if it is wealth-uniformly IRT for all $u \in U$.*

A natural benchmark class of U is

$$U_2 \equiv \{u : \mathbb{R} \to \mathbb{R} | u \text{ is strictly increasing and concave}\},$$

which corresponds to risk-averse expected utility maximizers. A U_2-DRT contract discourages risk-taking for all risk-averse agents at all wealth levels, while a U_2-IRT contract encourages risk-taking in the same sense.

The concept of DRT has appeared in the literature under different names. Ross (2004) refers to U-DRT contracts as *risk-inducing transformations*, while Hart (2011) studies a closely related notion based on *acceptance dominance*.[3] Both contributions focus on concave utility functions and impose additional structure on contracts, such as monotonicity or differentiability.

In particular, Ross (2004) establishes that a contract f is U-DRT if it "concavifies" utility, in the sense that $u(f(x))$ is a concave transformation of $u(x)$ for all $u \in U$.[4] While this condition is sufficient, it is not necessary. As we show in the next section, even simple additive contracts can be U_2-DRT without satisfying this concavification property. This observation highlights a gap between concavifying transformations and contracts that universally discourage risk-taking.

Our analysis builds on this insight by identifying a broader class of contracts that characterize decreasing and increasing risk-taking without requiring strong global restrictions such as monotonicity or concavity.

[3] In Ross (2004)'s definition, $u(w)$ on the right hand side of (1) is replaced by $u(w + f(0))$. Therefore, to cover both Ross' and Hart's definitions, $f(0) = 0$ needs to be imposed. We clarify a subtle role of $f(0) = 0$ in the Appendix B (Lemma B).

[4] $v(x)$ is a concave transformation of $u(x)$ if there is a concave function G such that $v(x) = G(u(x))$.

3 Additive Contracts

We begin by examining a simple and well-understood class of contracts that serves as a useful benchmark for our analysis. An *additive contract* is a function of the form $f(x) = \alpha x$, $\alpha > 0$. These contracts satisfy the additive property $f(x+y) = f(x) + f(y)$ for all x, y, and thus scale outcomes proportionally.[5] For $\alpha \neq 1$, additive contracts are regular in the sense of **Definition 3**.

Additive contracts provide a natural starting point because their effect on risk-taking is well known. In particular, when $\alpha > 1$, the contract amplifies both gains and losses, and is known to discourage risk-taking for risk-averse agents. We restate this result and provide a proof that highlights the key mechanism underlying our later analysis.

Proposition 1 (Gollier 2001, Hart 2011)[6]
An additive contract $f(x) = \alpha x$ with $\alpha > 1$ is U_2-DRT.

Proof. We show that for any $u \in U_2$, any w, and any $\alpha > 1$,

$$E[u(w+X)] \leq u(w) \;\; \Rightarrow \;\; E[u(w+\alpha X)] \leq u(w).$$

First, the identity $w + x = \frac{1}{\alpha}(w+\alpha x) + (1-\frac{1}{\alpha})w$ holds, where $w + x$ lies between $w + \alpha x$ and w, and $\frac{1}{\alpha} \in (0,1)$. By concavity of u,

$$u(w+x) \geq \frac{1}{\alpha} u(w+\alpha x) + \left(1 - \frac{1}{\alpha}\right) u(w).$$

Rearranging terms,

$$\alpha(u(w+x) - u(w)) \geq u(w+\alpha x) - u(w). \tag{3}$$

Taking expectations over a random variable X,

$$\alpha E[u(w+X) - u(w)] \geq E[u(w+\alpha X) - u(w)].$$

Since $\alpha > 1$, $E[u(w+X)] \leq u(w) \Rightarrow E[u(w+\alpha X)] \leq u(w)$. ■

The key inequality (3) in this proof arises directly from the concavity of u and does not rely on any comparison between the curvature of $u(w+\alpha x)$ and $u(w+x)$. This observation

[5] For the additive property, the absence of the constant term is critical. For this reason, we use “additive”. Carroll (2015) uses “affine” to allow for the constant and “linear” to exclude it.

[6] See p.632 in Hart (2011). Gollier (2001) also presents this result and its converse (if an additive contract with $\alpha > 1$ is U-DRT, then $U = U_2$) as Exercise 44 in Chapter 6, but does not provide its proof.

is important: it shows that decreasing risk-taking does not require $u(\alpha x)$ to be a concave transformation of $u(x)$.

This insight allows us to revisit a claim in the literature. Ross (2004) argues that a contract f is U-DRT *if and only if* it "concavifies" utility—that is, if $u(f(x))$ is a concave transformation of $u(x)$ for all $u \in U$.[7] While the "if" direction is correct, the "only if" direction is not. Additive contracts with $\alpha > 1$ provide a counterexample: they are U_2-DRT, yet need not concavify all concave utility functions. We formalize this observation in the following lemma.

Lemma 1. *For any $\alpha > 1$, there exists $u \in U_2$ such that the function $v(x) \equiv u(\alpha x)$ is not a concave transformation of $u(x)$.*

Sketch of Proof. Let $A_u(x) \equiv -\frac{u''(x)}{u'(x)}$ denote the Arrow–Pratt coefficient of absolute risk aversion. Recall that if v is a concave transformation of u, then $A_v(x) \geq A_u(x)$ for any x. For $v(x) \equiv u(\alpha x)$, $A_v(x) = \alpha A_u(\alpha x)$. Consider $u \in U_2$ that exhibits decreasing relative risk aversion over some interval containing positive values. Then for $0 < x < \alpha x$ on that interval, $x A_v(x) = \alpha x A_u(\alpha x) < x A_u(x)$, implying $A_v(x) < A_u(x)$. ■

This observation is not driven by pathological preferences. For example, it can arise under utility functions exhibiting decreasing absolute risk aversion; see Appendix B for a construction. Still, the result may appear counterintuitive. If v is not a concave transformation of u, one might expect that there exists a risk that is rejected under u but accepted under the transformed utility, contradicting the DRT property. However, two observations resolve this apparent paradox. First, the definition of DRT (or IRT) compares the rejection (or acceptance) of a given risk X with that of its transformed counterpart αX. If deviations in relative concavity occur only over gains (or only over losses), they may not affect the rejection (or acceptance) decision, since a risk consisting solely of gains (or losses) is never rejected (or accepted).

Second, even when such deviations occur over both gains and losses, the inequality (3) derived in the proof of **Proposition 1** ensures that the change in expected utility under the contract is bounded by the change without the contract, *up to a positive multiplicative factor*. This global inequality is sufficient to guarantee the DRT property, regardless of local curvature comparisons.

The analysis in this section highlights an important gap between concavifying transformations and contracts that decrease risk-taking. Additive contracts with $\alpha > 1$ belong to the

[7] See Theorem 4 in Ross (2004), p. 223. While the initial wealth w is not explicit in Ross (2004), it is immaterial for the current discussion because $u(w + \cdot)$ with different values of w can be viewed as different concave functions.

latter class but not necessarily the former. This suggests that the set of U_2-DRT contracts is substantially larger than previously recognized.

In the next section, we identify a broad class of contracts—single-crossing average (SCA) contracts—that generalizes additive contracts and fully captures this insight.

4 Single-Crossing Average Contracts

This section introduces a broad class of contracts that generalizes additive contracts and provides a sharp characterization of when contracts decrease or increase risk-taking. These are the single-crossing average (SCA) contracts.

The key idea is to impose a simple restriction on the average slope of the contract, measured by the ratio $\frac{f(x)}{x}$. This ratio compares the magnitude of the modified outcome to the original outcome and captures how gains and losses are transformed on average.

4.1 Definition and basic properties

We distinguish two types of SCA contracts, depending on whether they amplify or attenuate outcomes.

Definition 4 *A contract $f : \mathbb{R} \to \mathbb{R}$ is* ***single-crossing average from above*** *(write SCA^-) if:*

1. $f(0) = 0,\ \ f(x) \leq x$ *for all* $x < 0,\ \ f(x) \geq x$ *for all* $x > 0$*;*
2. *There exists* $\alpha > 1$ *such that* $\frac{f(x)}{x} \geq \alpha$ *for all* $x < 0$*,* $\frac{f(x)}{x} \leq \alpha$ *for all* $x > 0$*.*

Definition 5 *A contract $f : \mathbb{R} \to \mathbb{R}$ is* ***single-crossing average from below*** *(write SCA^+) if:*

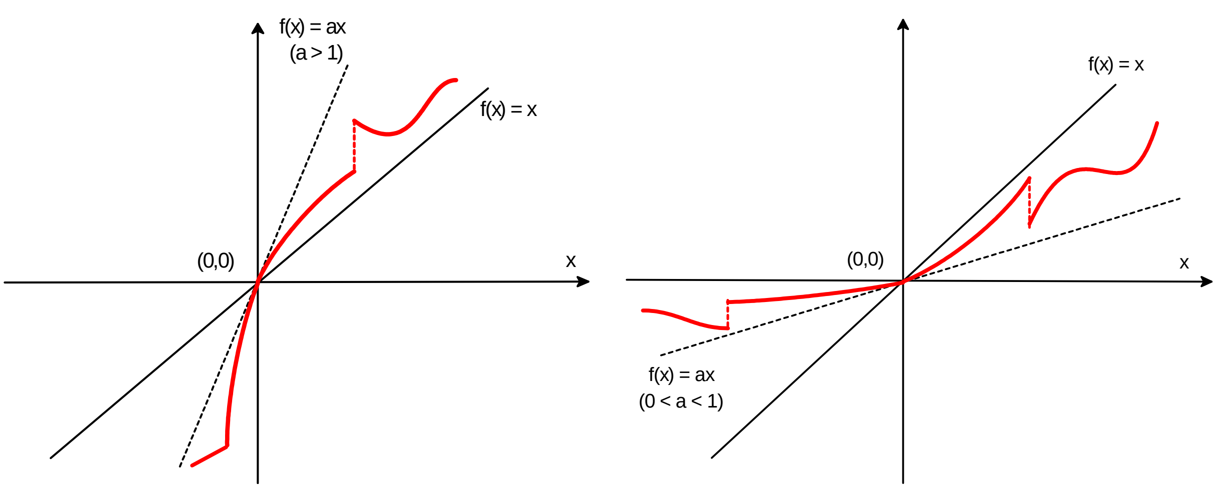
1. $f(0) = 0,\ \ 0 \geq f(x) \geq x$ *for all* $x < 0,\ \ f(x) \leq x$ *for all* $x > 0$*;*
2. *There exists* $\alpha \in (0, 1)$ *such that* $\frac{f(x)}{x} \leq \alpha$ *for all* $x < 0$*,* $\frac{f(x)}{x} \geq \alpha$ *for all* $x > 0$*.*

The conditions on positive and negative outcomes are stronger than regularity. They jointly ensure that the contract transforms gains and losses asymmetrically in a uniform direction. Let F^-_{SCA} and F^+_{SCA} denote the sets of SCA^- and SCA^+ contracts, respectively, and define $F_{SCA} \equiv F^-_{SCA} \cup F^+_{SCA}$. Additive contracts provide a special case: $f(x) = \alpha x$ with $\alpha > 1$ belongs to SCA^-, while $f(x) = \alpha x$ with $\alpha \in (0, 1)$ belongs to SCA^+.

Remark 1. In the definition of SCA contracts, the restriction $f(0) = 0$ is *not* without loss of generality. For broad classes of utility functions, any contract that uniformly decreases (or increases) risk-taking must satisfy $f(0) = 0$. Intuitively, a nonzero shift at zero can be exploited by suitably chosen utility functions to reverse the direction of the comparison. A formal result is provided in the Appendix B.

Remark 2. In the contract theory, a contract $f(x)$ is often formalized as a share of positive value x. From that perspective, SCA^- contracts (including additive contracts with $\alpha > 1$) may appear odd as they require $f(x) \geq x$ for $x > 0$. First, for our purpose, this is not a binding constraint on contracts. Given that a contract f decreases risk-taking, reducing $f(x)$ on the gain domain $x > 0$ does not destroy such property. However, recall that $f(x) \leq x$ for all x is a trivial contract that decreases risk-taking (hence we excluded it by the regularity condition). Second, contracts here can be viewed as modifications of existing contracts. Alternatively, the outcome x might be a market benchmark upon which contracts are written. Either way, $f(x) \geq x$ are not as strange as it may appear.

To build intuition, **Figure 1** illustrates representative examples of SCA^- and SCA^+ contracts. The figure highlights the single-crossing property of the average slope $\frac{f(x)}{x}$ at zero and the resulting asymmetry in the treatment of gains and losses.



Panel A. An SCA^- contract. **Panel B.** An SCA^+ contract.

Figure 1. **Single-crossing average (SCA) contracts.**

Note. In both panels, a solid curve (colored red) represents an SCA contract, and the dashed straight line represents an additive contract. The key feature is the single-crossing property of the average slope $\frac{f(x)}{x}$ at zero. **Panel A.** An SCA^{-} contract amplifies losses more than gains, thereby discouraging risk-taking. **Panel B.** An SCA^{+} contract attenuates losses more than gains, thereby encouraging risk-taking.

SCA contracts impose a *single-crossing property* on the average slope $\frac{f(x)}{x}$ at zero. For SCA^{-} contracts, the ratio $\frac{f(x)}{x}$ crosses a level $\alpha > 1$ from above as x moves through zero. These contracts amplify both gains and losses, but amplify losses more strongly in the following sense: $\frac{f(x)}{x}$ in the loss $x < 0$ is greater than $\frac{f(y)}{y}$ in the gain $y > 0$ *for any* $x < 0 < y$. For SCA^{+} contracts, the ratio $\frac{f(x)}{x}$ crosses a level $\alpha \in (0, 1)$ from below. These contracts attenuate both gains and losses, but attenuate losses more strongly in the similar "uniform" sense. This asymmetry between gains and losses is the key force driving changes in risk-taking behavior. Importantly, SCA contracts impose only a local restriction near zero. Away from zero, they may exhibit highly flexible behavior: they need not be monotonic or continuous, and can accommodate a wide range of economically relevant payoff structures.

4.2 Risk-taking implications

We now state the main result: SCA contracts characterize decreasing and increasing risk-taking for risk-averse agents.

Proposition 2 ***(SCA contracts and risk-taking).***

a. *Every* SCA^{-} *contract is* U_2*-DRT.*

b. *Every* SCA^{+} *contract is* U_2*-IRT.*

Proposition 2 shows that a purely local condition delivers a global comparative statics result. A key implication is that global concavity of the contract is not required to decrease risk-taking. The SCA condition is strictly weaker: it imposes a restriction on the average slope near zero rather than on the global curvature of f. This sharply contrasts with the concavification approach discussed in Section 3, and further highlights the gap between concavity-based restrictions on contracts.[8] Intuitively, SCA^{-} contracts discourage risk-taking because they increase losses more than gains in an average sense. Conversely, SCA^{+} contracts encourage risk-taking by reducing losses more than gains.

[8] To concavify all $u \in U_2$, $f''(x) \leq 0$ is necessary because the second derivative of $u(f(x))$ is $u''(f(x))(f'(x))^2 + u'(f(x))f''(x)$, and the first term is zero for the linear u. Ross (2004) shows that there is no f that concavifies all $u \in U_2$. For this approach, more restrictions on U_2 is necessary.

Proof of Proposition 2.

Any mass at zero can be handled separately by conditioning on $X \neq 0$, so without loss of generality assume $\Pr(X = 0) = 0$.

a. SCA^- **case (DRT)**

For any w and x such that $f(x) \neq 0$, the following identity holds:

$$w + x = \frac{x}{f(x)}(w + f(x)) + \left(1 - \frac{x}{f(x)}\right) w. \tag{4}$$

For $f \in F^-_{SCA}$, $\frac{x}{f(x)} \in (0, 1]$ and $w + x$ lies between $w + f(x)$ and w. By concavity of u,

$$u(w + x) \geq \frac{x}{f(x)} u(w + f(x)) + \left(1 - \frac{x}{f(x)}\right) u(w).$$

Rearranging terms,

$$\frac{f(x)}{x}(u(w + x) - u(w)) \geq u(w + f(x)) - u(w). \tag{5}$$

By the SCA^- property of f, there is $\alpha > 1$ such that for any $x \neq 0$, $\left(\alpha - \frac{f(x)}{x}\right)(u(w + x) - u(w)) \geq 0$. This is equivalent to

$$\alpha(u(w + x) - u(w)) \geq \frac{f(x)}{x}(u(w + x) - u(w)). \tag{6}$$

With the inequalities (5) and (6), taking expectations over a random variable X,

$$\alpha E[u(w + X) - u(w)] \geq E\left[\frac{f(X)}{X}(u(w + X) - u(w))\right] \geq E[u(w + f(X)) - u(w)].$$

Since $\alpha > 1$, $E[u(w + X)] \leq u(w) \Rightarrow E[u(w + f(X))] \leq u(w)$.

The proof for **b.** SCA^+ **case (IRT)** follows similar steps. See the Appendix A. ■

The proof follows three steps. First, for any x such that $\frac{x}{f(x)} \in (0, 1]$, the outcome $w + x$ can be expressed as a convex combination of w and $w + f(x)$ with the weight $\frac{x}{f(x)}$ attached to the latter. This is (4). Second, concavity of u allows utility changes by the modified outcome to be bounded above by utility changes by the original outcome scaled by $\frac{f(x)}{x} \geq 1$. This is (5). Finally, the SCA condition uniformly bounds the scaling factor $\frac{f(x)}{x}$ by a positive constant. This is (6). This yields a global comparison of expected utility changes.

Proposition 2 establishes that a local condition on the average slope $\frac{f(x)}{x}$ is sufficient to determine whether a contract discourages or encourages risk-taking for all risk-averse agents.

The driving force is an asymmetry between gains and losses: contracts that amplify losses more than gains discourage risk-taking, while those that attenuate losses more than gains encourage it. Notably, this conclusion does not require global concavity of the contract.

4.3 Closure properties under composition and addition

An important property of SCA contracts is that they are stable under composition. In many applications, contracts may be used to modify existing contracts. Hence contracts are layered: an initial contract transforms outcomes, and a subsequent contract further modifies the result. The composition $g \circ f \equiv g(f(x))$ captures this situation.

Lemma 2.

a. *If* $f, g \in F^{-}_{SCA}$, *then* $g \circ f \in F^{-}_{SCA}$.

b. *If* $f, g \in F^{+}_{SCA}$, *then* $g \circ f \in F^{+}_{SCA}$.

Lemma 2 shows that SCA contracts are closed under the composition transformations. As a result, their qualitative effect on risk-taking—whether discouraging or encouraging—is preserved even when multiple contractual layers are combined. This property enhances the practical relevance of the SCA class. A simple corollary of **Lemma 2** is that F^{-}_{SCA} is closed under multiplication of any constant $\alpha > 1$, while F^{+}_{SCA} is closed under multiplication of any constant $\alpha \in (0, 1)$. This follows trivially from the fact that additive contracts, when layered over SCA contracts, perform the associated scaler multiplications.

Addition of Contracts. A related question is whether SCA contracts are closed under addition. The answer differs across the two classes.

Lemma 3. *If* $f, g \in F^{-}_{SCA}$, *then* $f + g \in F^{-}_{SCA}$.

SCA^{-} contracts are closed under addition, whereas SCA^{+} contracts are not. The asymmetry reflects the fact that SCA^{+} contracts must attenuate losses, and this property may fail when two such contracts are offered together.[9] To encourage risk-taking by multiple SCA^{+} contracts, they should be layered, rather than added.

[9] $f(x) + g(x) < x < 0$ is possible even though $x < \min\{f(x), g(x)\} < 0$.

4.4 Characterization of risk aversion by SCA contracts

While SCA contracts decrease (or increase) risk-taking for all risk-averse agents, the reverse implication also holds: if all SCA contracts decrease (or increase) risk-taking, then the agent must be risk-averse.

We first prove a technical "if and only if" result.

Proposition 3. *Fix w and a utility function u.*

a. *$f \in F^{-}_{SCA}$ is DRT for u at w if and only if there exists $m > 1$ such that*

$$u(w + f(x)) - u(w) \leq m(u(w + x) - u(w)) \text{ for all } x. \tag{7}$$

b. *$f \in F^{+}_{SCA}$ is IRT for u at w if and only if there exists $m \in (0, 1)$ such that*

$$u(w + f(x)) - u(w) \geq m(u(w + x) - u(w)) \text{ for all } x. \tag{8}$$

Proposition 3 shows that DRT and IRT can be characterized by a global linear bound relating utility changes under the original and transformed outcomes. This result allows us to show the necessity of risk aversion under the assumption that u is twice-differentiable everywhere.

Proposition 4. *Suppose u is twice differentiable with $u'(x) > 0$ for all x.*

a. *If all $f \in F^{-}_{SCA}$ are wealth-uniformly DRT for u, then u is concave.*

b. *If all $f \in F^{+}_{SCA}$ are wealth-uniformly IRT for u, then u is concave.*

This result shows that SCA contracts provide a tight characterization of risk aversion. In particular, F^{-}_{SCA} and F^{+}_{SCA} are the mirror images to each other in terms of their implications for risk-taking.[10]

Remark 3. We briefly comment on the optimality of SCA contracts. First, for any SCA^{-} contract, there exists an additive contract in F^{-}_{SCA} that weakly increases the agent's expected utility (See Panel A in Figure 1). Second, for any SCA^{+} contract, there exists an additive contract in F^{+}_{SCA} that weakly decreases the agent's expected utility (See Panel B). This indicates that, if we impose the IRT property as a constraint on the set of contracts,

[10] From the definition of SCA contracts, if $f \in F^{-}_{SCA}$ is invertible, then $f^{-1} \in F^{+}_{SCA}$.

then additive contracts with $\alpha \in (0, 1)$ are a good candidate for the optimal contract. We leave this line of investigation for the future work.

We conclude this section with the following summary: SCA contracts form a large and tractable class that:

- Significantly generalizes additive contracts by requires only a single-crossing condition at zero on the average slope $\frac{f(x)}{x}$;
- Fully characterizes decreasing and increasing risk-taking for risk-averse agents;
- Is closed under composition and, for SCA^{-}, under addition.

These properties make SCA contracts a natural and powerful tool for analyzing how contractual transformations affect risk-taking behavior.

5 A Riskiness Index for Contracts

The analysis so far has identified broad classes of contracts that decrease or increase risk-taking for risk-averse agents. We now develop a local measure of contract riskiness that allows for a finer comparison across contracts. This index captures the behavior of a contract near zero that is crucial for risk-taking incentives and provides a complete ordering within a natural class of contracts.

When risks are small, the effect of a contract on risk-taking is governed by its local slope at zero. We therefore restrict attention to contracts that are twice-differentiable at zero and satisfy $f(0) = 0$. Let $f'(0)$ and $f''(0)$ denote the derivatives of f at zero.

5.1 Local characterization of U_2-DRT contracts

We first obtain the local characterization of U_2-DRT contracts, including (but not limited to) SCA contracts.

Lemma 4. *Regular U_2-DRT contracts f satisfy $f''(0) \leq 0 < f'(0) \neq 1$.*

Recall that the concavity of a contract f is not necessary for the U_2-DRT property. **Lemma 4** shows that the *local* concavity of f at $x = 0$ is a necessary condition for the U_2-DRT property. This condition is convenient to verify, because the local concavity at zero is much easier to check than the global concavity.

5.2 Index for contracts

For regular contracts such that $f'(0) \neq 1$ and $f''(0)$ exists, we define a contract index as follows.

Definition 6 *The contract index is* $I(f) \equiv \frac{f''(0)}{(f'(0)-1)f'(0)}$.

For example, additive contracts $f(x) = \alpha x$ with $\alpha \neq 1$ have $I(f) = 0$. For contracts with $f''(0) \neq 0 < f'(0)$, note that $f'(0) \gtrless 1$ matters for the sign of $I(f)$. The index $I(f)$ measures the local curvature of the contract relative to its local slope and summarizes how aggressively the contract distorts marginal risk around zero. It therefore plays a role analogous to the Arrow–Pratt coefficient of absolute risk aversion, but on the contract side rather than the preference side. The following characterization of regular contracts motivates the use of this index as a measure of contract riskiness.

Proposition 5. *Consider a regular contract f such that $f(0) = 0$ and a contract index $I(f)$ exists. If f is DRT for u at w, then either*

a. $f'(0) > 1$ *and* $A_u(w) \geq I(f)$, *or*

b. $f'(0) \in (0,1)$ *and* $A_u(w) \leq I(f)$.

If we want a regular contract f with $f'(0) > 1$ to discourage risk-taking by agent u with wealth level w, then the contract index $I(f)$ cannot exceed $A_u(w)$. If we want f to discourage risk-taking wealth-uniformly, then the upper bound of $I(f)$ becomes tighter to $\min_w A_u(w)$. Equivalently, for a given contract f with $f'(0) > 1$ and $I(f)$, the CARA agent with coefficient $I(f)$ is a benchmark agent: only agents more risk averse than the benchmark agent (in the Arrow-Pratt sense) can be wealth-uniformly discouraged by the contract f. The interpretation for a regular contract f with $f'(0) \in (0,1)$ is similar.[11]

Generally, SCA^- contracts that are twice-differentiable at zero satisfy $f'(0) > 1$ and $f''(0) \leq 0$. Therefore, the contract index $I(f)$ introduces a complete order among F^-_{SCA}: SCA^- contracts with the higher index are U-DRT for a smaller set U of utility functions. Therefore, SCA^- contracts with $I(f) < 0$ may discourage risk-taking by not only risk averse agents, but also some risk loving ones whose absolute risk aversion function $A_u(x)$ is uniformly bounded below by $I(f)$.

[11] The result for IRT is obtained by reversing the weak inequalities in Proposition 5.

Connection to the Acceptance Frontier. Yaari (1969) defined the acceptance frontier $f_{u,w}(x)$ for the agent with (u,w) by the following indifference condition:

$$pu(w+x)+(1-p)\,u(w+f)=u(w)\,.$$

Intuitively, let $\langle x,p,y\rangle$ be a binary gamble that pays off x with probability p and y with probability $1-p$. Then the agent with (u,w) is indifferent about accepting $\langle x,p,f_{u,w}(x)\rangle$, for any x. To see the connection to the contract index, differentiating the acceptance frontier twice yields

$$\begin{aligned} f'_{u,w}(x) &= -\frac{p}{1-p}\frac{u'(w+x)}{u'(w+f_{u,w}(x))}, \\ f''_{u,w}(x) &= -\frac{p}{1-p}\frac{u'(w+x)}{u'(w+f_{u,w}(x))}\left\{\frac{u''(w+x)}{u'(w+x)}-f'_{u,w}(x)\frac{u''(w+f_{u,w}(x))}{u'(w+f_{u,w}(x))}\right\}, \end{aligned}$$

so we have

$$\frac{f''_{u,w}(x)}{f'_{u,w}(x)}=\frac{u''(w+x)}{u'(w+x)}-f'_{u,w}(x)\frac{u''(w+f_{u,w}(x))}{u'(w+f_{u,w}(x))}.$$

Evaluating these at $x=0$, $f'_{u,w}(0)=-\frac{p}{1-p}<0$ and

$$\frac{f''_{u,w}(0)}{f'_{u,w}(0)}=-\frac{u''(w)}{u'(w)}\left(f'_{u,w}(0)-1\right)\Leftrightarrow A_u(w)=\frac{f''_{u,w}(0)}{f'_{u,w}(0)\left(f'_{u,w}(0)-1\right)}=I\left(f_{u,w}\right).$$

Thus, for the agent with (u,w), the contract index for the acceptance frontier is the agent's absolute risk aversion measure. Therefore, **Proposition 5** can be interpreted as comparing an *objective* contract f with a *subjective*, or hypothetical, contract $f_{u,w}$ that is indifferent (about accepting or not) to the agent. It is interesting that this comparison arises as *a necessary condition for DRT*, and that the direction of comparison depends on the local slope $f'(0)\gtrless 1$.

Applications. For a given set of contract F, we say that F is U-DRT if all $f\in F$ are U-DRT. Let $F^-\equiv\{f\in F|f'(0)>1,\ I(f)\ \text{exists}\}$ and $F^+\equiv\{f\in F|f'(0)\in(0,1)\,,\ I(f)\ \text{exists}\}$. The following result is immediate from **Proposition 5**.

Corollary.

a. *If F^- is U-DRT, then* $\min\limits_{u\in U}A_u(x)\geq\max\limits_{f\in F^-}I(f)$ *for all x.*

b. *If F^+ is U-DRT, then* $\max\limits_{u\in U}A_u(x)\leq\min\limits_{f\in F^+}I(f)$ *for all x.*

This Corollary suggests a simple way to quantify the effect of contracts on risk-taking. Suppose that F is a set of compensation schedules offered by a given firm to its managers. While utility functions of the managers are not observable, in principle contracts are observable so we can compute $\max_{f \in F^-} I(f)$ and $\min_{f \in F^+} I(f)$ for each firm. Let the computed value of $\max_{f \in F^-} I(f)$ for firm A by $I^-(A)$. If $I^-(A) \leq I^-(B)$, then we can interpret this as firm A being a more conservative towards its managers' risk-taking than firm B, because its compensation schedules are consistent with DRT *for a larger set of utility functions.*

6 Conclusion

This paper studies how contracts affect risk-taking by expected utility maximizers and provides a unified characterization of when contractual transformations discourage or encourage risk-taking. Our main result identifies single-crossing average (SCA) contracts as a broad and tractable class that fully captures these effects for risk-averse agents: contracts that increase losses more than gains on average discourage risk-taking, while those that decrease losses more than gains encourage it.

A key feature of SCA contracts is that they impose only a local restriction on the average slope $\frac{f(x)}{x}$, yet yield strong global implications. This characterization generalizes existing approaches based on concavity or monotonicity and clarifies the distinction between transformations that concavify utility and those that systematically alter risk-taking incentives. The class also exhibits useful structural properties: it is closed under composition, ensuring that its qualitative implications are preserved when contracts are layered.

We further introduce a local riskiness index, given by the derivatives at zero, which provides a complete ordering of contracts. This index complements the Arrow–Pratt measure of risk aversion and offers a simple way to quantify how contract design interacts with preferences.

The framework has natural applications in settings where contracts shape incentives, including executive compensation, asset management, and insurance design. More broadly, it shows that relatively weak, local restrictions on contracts can generate robust predictions about behavior. Extending this approach to non-expected utility models, dynamic environments, and empirical settings remains an important direction for future research.

7 Appendix A: Proofs

Proof of Lemma 2.

Let $h \equiv g \circ f$. $h(0) = g(f(0)) = 0$ holds for both F_{SCA}^{-} and F_{SCA}^{+}.

a. SCA^{-} **case.**

For F_{SCA}^{-}, $h(x) = g(f(x)) \le f(x) \le x$ for $x < 0$, $h(x) = g(f(x)) \ge f(x) \ge x$ for $x > 0$. Given $\alpha_f > 1$ such that $f(x) \le \alpha_f x$ $\forall x$ and $\alpha_g > 1$ such that $g(x) \le \alpha_g x$ $\forall x$, $\alpha_h \equiv \alpha_f \alpha_g > 1$ satisfies

$$\begin{aligned}\frac{h(x)}{x} &= \frac{g(f(x))}{x} = \frac{g(f(x))}{f(x)}\frac{f(x)}{x} \ge \alpha_f \alpha_g = \alpha_h \text{ for } x < 0,\\ \frac{h(x)}{x} &\le \alpha_f \alpha_g = \alpha_h \text{ for } x > 0.\end{aligned}$$

b. SCA^{+} **case.**

For F_{SCA}^{+}, $h(x) = g(f(x)) \ge f(x) \ge x$ and $g(f(x)) \le 0$ for $x < 0$, $h(x) = g(f(x)) \le f(x) \le x$ for $x > 0$. Given $\alpha_f \in (0,1)$ such that $f(x) \ge \alpha_f x$ $\forall x$ and $\alpha_g \in (0,1)$ such that $g(x) \ge \alpha_g x$ $\forall x$, $\alpha_h \equiv \alpha_f \alpha_g \in (0,1)$ satisfies

$$\begin{aligned}\frac{h(x)}{x} &= \frac{g(f(x))}{x} = \frac{g(f(x))}{f(x)}\frac{f(x)}{x} \le \alpha_f \alpha_g = \alpha_h \text{ for } x < 0,\\ \frac{h(x)}{x} &\ge \alpha_f \alpha_g = \alpha_h \text{ for } x > 0. \quad \blacksquare\end{aligned}$$

Proof of Lemma 3.

Let $h \equiv f + g$. Again, $h(0) = f(0) + g(0) = 0$ holds for F_{SCA}^{-}.

Next, $h(x) = f(x) + g(x) \le 2x < x$ for $x < 0$, $h(x) = f(x) + g(x) \ge 2x > x$ for $x > 0$. Given $\alpha_f > 1$ such that $f(x) \le \alpha_f x$ $\forall x$ and $\alpha_g > 1$ such that $g(x) \le \alpha_g x$ $\forall x$, $\alpha_h \equiv \alpha_f + \alpha_g > 2$ satisfies

$$\begin{aligned}\frac{h(x)}{x} &= \frac{f(x)}{x} + \frac{g(x)}{x} \ge \alpha_h \text{ for } x < 0,\\ \frac{h(x)}{x} &\le \alpha_h \text{ for } x > 0. \quad \blacksquare\end{aligned}$$

Proof of Proposition 2b (SCA^{+} case).

For any w and $x \ne 0$, the identity $w + f(x) = \frac{f(x)}{x}(w + x) + \left(1 - \frac{f(x)}{x}\right) w$ holds. For $f \in F_{SCA}^{+}$, $\frac{f(x)}{x} \in (0,1]$ and $w + f(x)$ lies between $w + x$ and w. By concavity of u, $u(w + f(x)) \ge \frac{f(x)}{x} u(w + x) + \left(1 - \frac{f(x)}{x}\right) u(w)$. Rearranging terms, $u(w + f(x)) - u(w) \ge$

$\frac{f(x)}{x}\left(u\left(w+x\right)-u\left(w\right)\right)$. By the SCA^{+} property of f, there is $\alpha \in (0,1)$ such that for any $x \neq 0$, $\frac{f(x)}{x}\left(u\left(w+x\right)-u\left(w\right)\right) \geq \alpha\left(u\left(w+x\right)-u\left(w\right)\right)$. Therefore,

$$E\left[u\left(w+f\left(X\right)\right)-u\left(w\right)\right] \geq E\left[\frac{f\left(X\right)}{X}\left(u\left(w+X\right)-u\left(w\right)\right)\right] \geq \alpha E\left[u\left(w+X\right)-u\left(w\right)\right].$$

Since $\alpha \in (0,1)$, $E\left[u\left(w+X\right)\right] \geq u\left(w\right) \Rightarrow E\left[u\left(w+f\left(X\right)\right)\right] \geq u\left(w\right)$. ■

Proof of Proposition 3.

The proof uses two-point risks to translate the DRT/IRT condition into a pointwise bound. Let $g_1\left(x\right) \equiv u\left(w+x\right)-u\left(w\right)$ and $g_2\left(x\right) \equiv u\left(w+f\left(x\right)\right)-u\left(w\right)$. Since u is strictly increasing, $g_1\left(x\right)$ has the same sign as x. The DRT condition (1) is

$$E\left[g_1\left(X\right)\right] \leq 0 \;\Rightarrow\; E\left[g_2\left(X\right)\right] \leq 0. \tag{9}$$

a. SCA^{-} case (DRT).

[If] Suppose there exists $m > 1$ such that $g_2\left(x\right) \leq m g_1\left(x\right)\ \forall x$. Taking expectations yields $E\left[g_2\left(X\right)\right] \leq mE\left[g_1\left(X\right)\right]$. Hence (9) holds.

[Only if] We proceed in three steps:

1) Derive (7) without any restriction on $m \in \mathbb{R}$.

2) Show $m > 0$ and $m \neq 1$ are necessary for regular contracts.

3) Show $m > 1$ is necessary for SCA^{-} contracts.

Step 1. Consider a random variable

$$X = \begin{cases} x_1 & \text{with probability } p \\ x_2 & \text{with probability } 1-p \end{cases}, \quad x_1 < 0 < x_2.$$

Because $g_1\left(x_1\right) < 0 < g_1\left(x_2\right)$, we set $p = \frac{g_1(x_2)}{g_1(x_2)-g_1(x_1)} \in (0,1)$ so that $E\left[g_1\left(X\right)\right] = 0$. By (9), $E\left[g_2\left(X\right)\right] \leq 0$ holds for this X. Therefore,

$$\frac{g_1\left(x_2\right)}{g_1\left(x_2\right)-g_1\left(x_1\right)} g_2\left(x_1\right) + \left(1-\frac{g_1\left(x_2\right)}{g_1\left(x_2\right)-g_1\left(x_1\right)}\right) g_2\left(x_2\right) \leq 0 \;\Leftrightarrow\; \frac{g_2\left(x_2\right)}{g_1\left(x_2\right)} \leq \frac{g_2\left(x_1\right)}{g_1\left(x_1\right)}.$$

Because this must hold for any $x_1 < 0 < x_2$, there exists some $m \in \mathbb{R}$ such that $g_2\left(x\right) \leq m g_1\left(x\right)\ \forall x$ given $g_1\left(x\right) \neq 0$. Finally, (9) for a random variable X that puts probability one to 0 implies $E\left[g_1\left(X\right)\right] = g_1\left(0\right) = 0$ and $E\left[g_2\left(X\right)\right] = g_2\left(0\right) = 0$. Therefore, $g_2\left(x\right) \leq m g_1\left(x\right)$ holds for any x. This is (7).

Step 2. Suppose $m \leq 0$ in (7). Then $\frac{g_2(x)}{g_1(x)} \leq m \leq 0$ for any $x > 0$ implies that $g_2\left(x\right) \leq 0 \Leftrightarrow f\left(x\right) \leq 0$ for any $x > 0$, contradicting the regularity of f. Also, $m = 1$ in (7) implies $f(x) \leq x$ for all x, contradicting the regularity.

Step 3. Rewrite (7) with $m > 0$ as

$$\left(1 - \frac{1}{m}\right) u(w) + \frac{1}{m} u(w + f(x)) \leq u(w + x). \tag{10}$$

The left hand side of (10) is $u(w + f(x)) + \left(\frac{1}{m} - 1\right)(u(w + f(x)) - u(w))$. If $m < 1$ and $u(w + f(x)) - u(w) > 0$, then (10) cannot hold. Because SCA^- contract f must satisfy $u(w) < u(w + x) \leq u(w + f(x))$ for $x > 0$, $m < 1$ cannot be true.

b. SCA^+ case (IRT).

The argument is symmetric. The IRT condition (2) is

$$E[g_1(X)] \geq 0 \;\Rightarrow\; E[g_2(X)] \geq 0. \tag{11}$$

[If] If there is $m \in (0, 1)$ such that $g_2(x) \geq m g_1(x)\ \forall x$, taking expectations yields $E[g_2(X)] \geq m E[g_1(X)]$. Hence (11) holds.

[Only if]

Step 1. Consider the same random variable X as defined above, which satisfies $E[g_1(X)] = 0$. By (11), $E[g_2(X)] \geq 0$ holds for this X. Therefore, $\frac{g_2(x_2)}{g_1(x_2)} \geq \frac{g_2(x_1)}{g_1(x_1)}$ for any $x_1 < 0 < x_2$. Then there exists some $m \in \mathbb{R}$ such that $g_2(x) \geq m g_1(x)\ \forall x$ such that $g_1(x) \neq 0$. Finally, (11) for a random variable X that puts probability one to 0 implies $E[g_1(X)] = g_1(0) = 0$ and $E[g_2(X)] = g_2(0) = 0$. Therefore, $g_2(x) \geq m g_1(x)$ holds for any x. This is (8).

Step 2. Suppose $m \leq 0$ in (8). Then $\frac{g_2(x)}{g_1(x)} \leq m \leq 0$ for any $x < 0$ implies that $g_2(x) \geq 0 \Leftrightarrow f(x) \geq 0$ for any $x < 0$, contradicting the regularity of f. Also, $m = 1$ in (8) implies $f(x) \geq x$ for all x, contradicting the regularity.

Step 3. We want to show $m < 1$. Rewrite (8) with $m > 0$ as $\left(1 - \frac{1}{m}\right) u(w) + \frac{1}{m} u(w + f(x)) \geq u(w + x)$. If $m > 1$, then this contradicts the fact that any $f \in F^+_{SCA}$ satisfies $u(w) < u(w + f(x)) \leq u(w + x)$ for $x > 0$. ■

Proof of Proposition 4.

a. SCA^- case (DRT).

Define $H(x) \equiv u(w + f(x)) - u(w) - m(u(w + x) - u(w))$ to write (7) as $H(x) \leq 0$ for any x. Because $f(0) = 0$ implies $H(0) = 0$, a necessary condition is that $H(x) \leq 0$ holds in the neighborhood of $x = 0$. Therefore, we evaluate a local necessary condition

$H'(0) = 0 \geq H''(0)$ for $f(x) = \alpha x$ with $\alpha > 1$, which is a differentiable SCA^- contract.

$$\begin{aligned} H'(x) &= u'(w + f(x)) f'(x) - mu'(w + x), \\ H''(x) &= u''(w + f(x)) (f'(x))^2 + u'(w + f(x)) f''(x) - mu''(w + x). \end{aligned}$$

Using $f(0) = 0$, $f'(x) = \alpha$, and $f''(x) = 0$,

$$\begin{aligned} H'(0) &= u'(w) \alpha - mu'(w) = 0 \Leftrightarrow m = \alpha, \\ H''(0) &= u''(w) \alpha^2 - \alpha u''(w) = u''(w) \alpha (\alpha - 1) \leq 0. \end{aligned}$$

Then $\alpha > 1$ implies $u''(w) \leq 0$. Because this must hold for any w, u must be concave.

b. SCA^+ case (IRT).

Using the same $H(x)$ as above, (8) is $H(x) \geq 0$ for any x. We evaluate a local necessary condition $H'(0) = 0 \leq H''(0)$ for $f(x) = \alpha x$ with $\alpha \in (0, 1)$. So $m = \alpha$ and $H''(0) = u''(w) \alpha (\alpha - 1) \geq 0$. Because $\alpha \in (0, 1)$, $u''(w) \leq 0$. ■

Remark. We assumed differentiability of utility functions to prove **Proposition 4**, but we did not impose differentiability for SCA contracts. The necessity of risk aversion for risk-taking to be discouraged by *all* SCA contracts was established by checking additive contracts, which are differentiable SCA contracts. If we consider a subset of SCA contracts that do not include additive contracts, risk aversion may not be necessary. We conjecture that the necessity of risk aversion can be proved without resorting additive contracts and/or differentiability of utility functions. A proof without relying on additive contracts or differentiability remains an open question.

Proof of Lemma 4 and Proposition 5.

We use the following result, which does not impose $f(0) = 0$.

Lemma A. *Suppose u is twice differentiable with $u'(x) > 0$ for all x, and that $f(x)$ is twice differentiable at $x = 0$. Then a regular contract f that is DRT for u at w satisfies:*

$$(7) \text{ with } m = f'(0) \frac{u'(w + f(0))}{u'(w)} > 0,\ m \neq 1,$$

and

$$A_u(w) - f'(0) A_u(w + f(0)) \leq A_f(0). \tag{12}$$

Proof of Lemma 4.

Because U_2 contains linear utility functions, $\frac{u'(w+f(0))}{u'(w)} = 1$ implies $m = f'(0)$. Substituting $A_u(w) = A_u(w + f(0)) = 0$ into (12), $0 \leq A_f(0) \Leftrightarrow f''(0) \leq 0$. ■

Proof of Proposition 5.

Imposing $f(0) = 0$ in **Lemma A** yields $m = f'(0)$ and $(1 - f'(0)) A_u(w) \leq A_f(0)$. In the proof of **Proposition 3** (Step 2), it was proved that $m > 0$ and $m \neq 1$ hold for regular contracts. Therefore,

$$\begin{cases} A_u(w) \geq \frac{f''(0)}{(f'(0)-1)f'(0)} = I(f) & \text{for } f'(0) > 1, \\ A_u(w) \leq \frac{f''(0)}{(f'(0)-1)f'(0)} = I(f) & \text{for } f'(0) \in (0,1). \end{cases}$$ ■

Proof of Lemma A.

A necessary condition for $H(x) \leq 0 \ \forall x$ is

$$H'(0) = 0 \geq H''(0), \tag{13}$$

where $H(x) \equiv u(w + f(x)) - u(w) - m(u(w + x) - u(w))$. First,

$$\begin{aligned} H'(x; m) &= u'(w + f(x)) f'(x) - mu'(w + x), \\ H''(x; m) &= u''(w + f(x)) (f'(x))^2 + u'(w + f(x)) f''(x) - mu''(w + x). \end{aligned}$$

Given the differentiability assumptions on (u, f),

$$\begin{aligned} H'(0) &= u'(w + f(0)) f'(0) - mu'(w), \\ H''(0) &= u''(w + f(0)) (f'(0))^2 + u'(w + f(0)) f''(0) - mu''(w). \end{aligned}$$

Therefore, a necessary condition (13) is $m = f'(0) \frac{u'(w+f(0))}{u'(w)}$ and

$$u''(w + f(0)) (f'(0))^2 + u'(w + f(0)) f''(0) \leq f'(0) \frac{u'(w + f(0))}{u'(w)} u''(w). \tag{14}$$

Because $m > 0$ holds for regular DRT contracts (see Step 2 in the proof of **Proposition 3**), $f'(0) > 0$. Then (14) can be written as $\frac{u''(w+f(0))}{u'(w+f(0))} f'(0) + \frac{f''(0)}{f'(0)} \leq \frac{u''(w)}{u'(w)}$. Using $A_u(x) \equiv -\frac{u''(x)}{u'(x)}$ and $A_f(x) \equiv -\frac{f''(x)}{f'(x)}$, this is (12). ■

8 Appendix B: Additional Results

8.1 DARA example

Let a utility function $u(x)$ have $A_u(x) = \frac{1}{x^{1+\epsilon}}$, $\epsilon > 0$, on some interval $I = [x_1, x_2]$, $x_1 > 0$, so that it exhibits decreasing relative risk aversion on I. This utility function may also exhibit globally decreasing absolute risk aversion (DARA). For example, consider $A_u(x) = 1$ for $x \leq 1$ and $A_u(x) = \frac{1}{x^{1+\epsilon}}$, $\epsilon > 0$ for $x > 1$. To the extent that DARA utility functions are not pathological, neither is the gap between U_2-DRT contracts and concavifying contracts.

8.2 Necessity of $f(0) = 0$

The next result makes precise the properties of u that demand $f(0) = 0$.

Lemma B. *Suppose that, for any $c > 0$, $\exists u \in U$ such that $\inf_{x>0} \frac{u'(x+c)}{u'(x)} = 0$ and also $v \in U$ such that $\sup_{x>0} \frac{v'(x-c)}{v'(x)} = \infty$. Then any regular U-DRT contract satisfies $f(0) = 0$.*

Proof. By **Lemma A**, the DRT condition (7) with $m = f'(0) \frac{u'(w+f(0))}{u'(w)}$ is

$$u(w + f(x)) - u(w) \leq f'(0) \frac{u'(w + f(0))}{u'(w)} (u(w + x) - u(w)) \ \forall x.$$

If $f(0) \neq 0$, one can choose utility functions with sufficiently extreme marginal utility ratios so that the above inequality fails. Suppose $f(0) > 0$. By regularity, there is $x > 0$ such that both sides are positive. By choosing u and w such that $\frac{u'(w+f(0))}{u'(w)}$ is close to zero, the inequality is violated. The case for $f(0) < 0$ is similar. ■

Lemma B shows the necessity of $f(0) = 0$ for U_2-DRT contracts. Any differentiable $u \in U_2$ satisfies $\frac{u'(y)}{u'(x)} \in (0, 1]$ for any $x < y$. However, for arbitrary (x, y), $\frac{u'(y)}{u'(x)}$ may be close to zero (for $x < y$) or infinity (for $x > y$). This suggests that if contracts with $f(0) \neq 0$ are DRT-U, then U must be strictly smaller than U_2. In particular, $u'(x)$ must be uniformly and strictly bounded away from zero.

We show that $\forall u \in U_2$, $\forall w$, $\forall f \in F^-_{SCA}$,

$$E[u(w + X)] \leq u(w) \ \Rightarrow \ E[u(w + f(X))] \leq u(w). \tag{15}$$

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