Technical Report:
Limited Liability and the Inter-Bank Loan Trading*

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1 Introduction

We present a model of inter-bank market where heterogeneous banks trade risky asset. Unlike most models in the banking literature, we assume that banks are risk-averse. Another key feature of the model is that limited liability induces relatively more efficient banks to endogenously borrow and take additional risk in the inter-bank market. We prove some preliminary results and discuss other aspects of the model to be investigated.

2 Model

There is a continuum of banks. We assume that all banks have a mean-variance preference $E[\pi] - \frac{\gamma}{2}Var[\pi]$ over their profit π, where $\gamma > 0$ is a risk-aversion parameter. There are three stages, ex ante, interim and ex post. Banks act at the ex ante stage and the ex post stage. At the ex ante stage, all banks are homogenous. They start with the same amount

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of safe asset $A > 0$ with return R_0. They choose (identical) investment V with a stochastic return $\widetilde{R}$ per unit (e.g. lending) and a safe asset position $W = A - V$ with a return R_0 per unit (e.g. short-term government debt). At the interim stage, each bank independently draws a high success probability p_H of the risky asset with probability $\lambda \in (0, 1)$, or a low success probability $p_L < p_H$ with probability $1 - \lambda$. By the law of large numbers, a measure λ of banks have a high success probability. For type $i \in \{L, H\}$ banks, $\widetilde{R} = R$ with probability p_i and $\widetilde{R} = 0$ with probability $1 - p_i$. Banks with p_H are called *H-banks* and banks with p_L are called *L-banks*. Let $\omega_i \equiv p_i (1 - p_i)$ for $i \in \{H, L\}$.

Assumption 1 $\frac{1}{2} \leq p_L < p_H < 1$.

Assumption 1 implies $0 < \omega_H < \omega_L \leq \frac{1}{4}$, which means that H-banks face a higher mean return $E\left[\widetilde{R}\right] = p_H R$ as well as a lower variance of return $Var\left[\widetilde{R}\right] = \omega_H R^2$. Therefore, type H is a better type. This heterogeneity leads to ex post trading of risky loan between L-banks and H-banks.

At the ex post stage, banks can trade a unit of risky asset with Q units of safe asset. Q will be determined in a market-clearing equilibrium. When choosing V at the ex ante stage, banks rationally anticipate what will happen at the interim and the ex post stage.

In Section 3, we study banks' ex post trading problem for a given price Q. We explain how problem changes depending on whether H-banks borrow or not. In Section 4, we define two types of market-clearing equilibria, and explain how they might coexist (i.e., equilibrium multiplicity). We also explain banks' ex ante investment problem. In Section 5, we list remaining tasks to be done.

3 Ex post trading

At the ex post stage, banks take $V < A$ as given, and trade the risky asset to change their position from V to $k + V$. Positive $k > 0$ corresponds to a purchase, while $k < 0$ corresponds to a sales. We assume that banks can borrow a safe asset at rate $B > R_0$ up to the amount that can be used for the purchase of risky asset, i.e., $Qk > 0$. Banks' risk profile changes depending on their choice of borrowing. We first consider a case where banks do not borrow. In the later subsection we study how borrowing changes banks' behavior.

3.1 Optimal trading without borrowing

We first consider a case where the purchase of risky asset is within the initial safe asset position, i.e., $Qk \leq A - V$. A profit for type i banks is

$$\pi_i(k) = \begin{cases} R(V+k) + (A - V - Qk) R_0 & \text{with probability } p_i \\ (A - V - Qk) R_0 & \text{with probability } 1 - p_i. \end{cases}$$

By limiting the ex post risky asset position up to $\frac{A-V}{Q} + V = \frac{A+(Q-1)V}{Q}$, banks can secure the minimum profit of $(A - V - Qk) R_0 \geq 0$ independent of their types. Therefore, we define *non-risky profit* by

$$\pi^{NR}(k;Q) \equiv (A - V - Qk) R_0.$$

For $k \leq \frac{A-V}{Q}$, $\pi^{NR}(k;Q)$ is non-negative and decreases in k. Using $\pi^{NR}(k;Q) = \{A + (Q-1)V\} R_0 - (V+k) QR_0$, a mean and variance of $\pi_i(k)$ is

$$\begin{aligned} E_i[\pi_i(k)] &= p_i R(V+k) + \pi^{NR}(k;Q) \\ &= (p_i R - QR_0)(V+k) + \{A + (Q-1)V\} R_0, \\ V_i[\pi_i(k)] &= \omega_i R^2 (V+k)^2. \end{aligned}$$

The associated payoff is

$$U_i(k;Q) = (p_i R - QR_0)(V+k) - \frac{\gamma}{2}\omega_i R^2 (V+k)^2 + \{A + (Q-1)V\} R_0. \tag{1}$$

Taking a price Q given, banks choose k to maximize (1). Ignoring a term independent of k, and noting that a risky asset position $V + k$ cannot be negative (i.e., a short-selling constraint), banks solve

$$\max_k \left\{ (p_i R - QR_0)(V+k) - \frac{\gamma}{2}\omega_i R^2 (V+k)^2 \right\}$$

$$\text{subject to } -V \leq k \leq \frac{A-V}{Q}.$$

For trading to occur, chosen k must be positive for some banks and negative for other banks. Given **Assumption 1**, the short-selling constraint $-V \leq k$ is relevant only for L-banks while the borrowing constraint $k \leq \frac{A-V}{Q}$ is relevant only for H-banks. Therefore, a solution to this problem is the following optimal order $k_i^*(Q)$ for $i \in \{L, H\}$:

$$k_L^*(Q) \equiv \max\left\{ \frac{p_L - Q\frac{R_0}{R}}{\gamma R \omega_L} - V, -V \right\} \text{ and } k_H^*(Q) \equiv \min\left\{ \frac{p_H - Q\frac{R_0}{R}}{\gamma R \omega_H} - V, \frac{A-V}{Q} \right\} \quad (2)$$

For any given Q, we denote by $U_i^*(Q) \equiv U_i(k_i^*(Q); Q)$ the payoff for type i banks submitting (2). When the order (2) is unconstrained (i.e., $k_i^*(Q) = \frac{p_i - Q\frac{R_0}{R}}{\gamma R\omega_i} - V \in \left[0, \frac{A-V}{Q}\right]$),

$$U_i^*(Q) = U_i\left(\frac{p_i - Q\frac{R_0}{R}}{\gamma R \omega_i}; Q \right) = \{A + (Q-1)V\} R_0 + \frac{1}{2\gamma}\frac{p_i}{1-p_i}\left(\frac{p_i R - QR_0}{p_i R}\right)^2. \quad (3)$$

When L-banks are constrained (i.e., $p_L - Q\frac{R_0}{R} < 0$),

$$U_L^*(Q) = U_L(-V; Q) = \pi^{NR}(-V; Q) = \{A + (Q-1)V\} R_0.$$

When H-banks are constrained (i.e., $\frac{p_H - Q\frac{R_0}{R}}{\gamma R \omega_H} > \frac{A-V}{Q} + V$),

$$U_H^*(Q) = U_H\left(\frac{A-V}{Q}; Q\right) = \frac{p_H R}{Q}\{A + (Q-1)V\} - \frac{\gamma}{2}\frac{1-p_H}{p_H}\left[\frac{p_H R}{Q}\{A + (Q-1)V\}\right]^2.$$

In sum, the payoff after trading without borrowing is

$$U_i^*(Q) = \begin{cases} \{A+(Q-1)V\}R_0 & \text{for } i \in \{L,H\} & \text{if } \begin{array}{l} Q\frac{R_0}{R} \leq p_L \text{ and} \\ \frac{p_H - Q\frac{R_0}{R}}{\gamma R \omega_H} \leq \frac{A-V}{Q} + V, \end{array} \\ \quad +\frac{1}{2\gamma}\frac{p_i}{1-p_i}\left(\frac{p_i R - QR_0}{p_i R}\right)^2 & & \\ \{A+(Q-1)V\}R_0 & \text{for } i = L & \text{if } Q\frac{R_0}{R} > p_L, \\ \{A+(Q-1)V\}\frac{p_H R}{Q} & \text{for } i = H & \text{if } \frac{p_H - Q\frac{R_0}{R}}{\gamma R \omega_H} > \frac{A-V}{Q} + V. \\ \quad -\frac{\gamma}{2}\frac{1-p_H}{p_H}\left[\{A+(Q-1)V\}\frac{p_H R}{Q}\right]^2 & & \end{cases} \tag{4}$$

3.2 Optimal trading with borrowing

Next, we consider a case where banks borrow at rate $B > R_0$ to fund the purchase of risky asset. We assume that banks can borrow at most Qk, and consider two cases. First, $Qk \in (0, A-V]$ such that borrowing is not necessary. We will show below that borrowing may occur even in this case because of limited liability. Second, $Qk > A - V$ such that borrowing is absolutely necessary to achieve a risky asset position $k + V$.

Borrow and buy $Qk \in (0, A-V]$. Banks do not need to borrow to buy $k \in \left(0, \frac{A-V}{Q}\right]$ units of risky asset, but they still can. Let $X \in [0, Qk]$ be the amount borrowed. H-banks' ex post safe asset position becomes $A - V - (Qk - X)$. A profit for H-banks is

$$\widehat{\pi}_H\left(k \leq \frac{A-V}{Q}, X\right) = \begin{cases} R(V+k) + R_0\{A-V-(Qk-X)\} - BX & \text{with probability } p_H \\ [R_0\{A-V-(Qk-X)\} - BX]^+ & \text{with probability } 1-p_H, \end{cases}$$

where $[R_0\{A-V-(Qk-X)\} - BX]^+$ is a positive part of $R_0\{A-V-(Qk-X)\} - BX$. For the choice of $X \in [0, Qk]$, notice that if $R_0\{A-V-(Qk-X)\} - BX = R_0(A-V-Qk) - (B-R_0)X$ is strictly positive, by reducing X, H-banks can increase the mean of their profit without affecting its variance. Therefore, whenever possible, they choose X such that $[R_0\{A-V-(Qk-X)\}-BX]^+=0$. That is,

$$X \geq \frac{R_0}{B-R_0}(A-V-Qk) \equiv \underline{X}(k). \tag{5}$$

For such a choice of X to be feasible, we need $\underline{X}(k) \leq Qk \Leftrightarrow k \geq \frac{R_0}{B}\frac{A-V}{Q}$. For a given $k \in \left[\frac{R_0}{B}\frac{A-V}{Q}, \frac{A-V}{Q}\right]$, $\underline{X}(k)$ defined in (5) is the minimum amount of borrowing that makes $[R_0\{A-V-(Qk-X)\}-BX]^+=0$. On the (k,X)-plane, two lines $X=Qk$ and $X=\underline{X}(k)$ have a unique intersection $(k,X)=\left(\frac{R_0}{B}\frac{A-V}{Q}, \frac{R_0(A-V)}{B}\right)$. Hence,

$$\left\{(k,X)\,|X \in [\underline{X}(k), Qk], k \in \left[\frac{R_0}{B}\frac{A-V}{Q}, \frac{A-V}{Q}\right]\right\}$$

forms a triangular region on the (k,X)-plane.

Denoting the success profit by

$$\Gamma_1(k,X) \equiv (R-R_0Q)k-(B-R_0)X+R_0A+(R-R_0)V,$$

H-banks' problem can be written as

$$\max_{(k,X)}\left\{p_H\Gamma_1(k,X)-\frac{\gamma}{2}\omega_H(\Gamma_1(k,X))^2\right\} \tag{6}$$

$$\text{subject to } (k,X) \in \left\{(k,X)\,|X \in [\underline{X}(k), Qk], k \in \left[\frac{R_0}{B}\frac{A-V}{Q}, \frac{A-V}{Q}\right]\right\}.$$

We observe two facts. First, given $R > R_0Q$, $\Gamma_1(k,X)$ is increasing in k and decreasing in X. Second, (6) is maximized when $\Gamma_1(k,X)=\frac{1}{\gamma}\frac{1}{1-p_H}$ and achieves

$$\frac{1}{2\gamma}\frac{p_H}{1-p_H} \equiv \widehat{U}_H,$$

which is independent of (Q,V,A,B,R,R_0). We define the "iso-payoff" line on the (k,X)-

plane by $\left\{(k,X)\,|\Gamma_1(k,X)=\frac{1}{\gamma}\frac{1}{1-p_H}\right\}$. This is linear in (k,X):

$$X=\frac{R-R_0Q}{B-R_0}k-\frac{R}{B-R_0}\left\{\frac{1}{\gamma R}\frac{1}{1-p_H}-V-\frac{R_0(A-V)}{R}\right\}\equiv X_1^{IP}(k). \tag{7}$$

Solutions to the problem (6) are characterized as a segment of the iso-payoff line (7) inside the triangular area $\left\{(k,X)\,|X\in\left[\underline{X}(k),Qk\right],k\in\left[\frac{R_0}{B}\frac{A-V}{Q},\frac{A-V}{Q}\right]\right\}$. Importantly, both $\underline{X}(k)$ and $X_1^{IP}(k)$ increase in A, but they have a unique intersection that does not depend on A:

$$\underline{X}(k)=X_1^{IP}(k)\Leftrightarrow k=\frac{1}{\gamma R}\frac{1}{1-p_H}-V\equiv k^{LB}. \tag{8}$$

For the remaining analysis, we proceed under a conjecture $\frac{R-R_0Q}{B-R_0}>Q>0\Leftrightarrow\frac{R}{Q}>B>R_0$. This needs to be verified in any equilibrium. In particular, $\frac{R-R_0Q}{B-R_0}>Q$ implies that the iso-payoff line (7) goes through the triangular area if and only if

$$A\in\left[\overline{A}(Q),\overline{A}\left(\frac{BQ}{R_0}\right)\right].$$

The lower bound $\overline{A}(Q)$ is a value of A that solves $\underline{X}\left(\frac{A-V}{Q}\right)=X_1^{IP}\left(\frac{A-V}{Q}\right)$ and is given by

$$\overline{A}(Q)\equiv Q\frac{1}{\gamma R}\frac{1}{1-p_H}+(1-Q)V=V+Qk^{LB}. \tag{9}$$

The upper bound $\overline{A}\left(\frac{BQ}{R_0}\right)$ solves $\underline{X}\left(\frac{R_0}{B}\frac{A-V}{Q}\right)=X_1^{IP}\left(\frac{R_0}{B}\frac{A-V}{Q}\right)$ and is given by

$$\overline{A}\left(\frac{BQ}{R_0}\right)=\frac{BQ}{R_0}\frac{1}{\gamma R}\frac{1}{1-p_H}+\left(1-\frac{BQ}{R_0}\right)V=V+\frac{BQ}{R_0}k^{LB}. \tag{10}$$

Given $B>R_0$, $\overline{A}(Q)<\overline{A}\left(\frac{BQ}{R_0}\right)$ holds if and only if $k^{LB}>0$. Also,

$$A<\overline{A}\left(\frac{BQ}{R_0}\right)\Leftrightarrow A-V<\frac{BQ}{R_0}k^{LB}.$$

Therefore, for any given $A - V > 0$, $A < \overline{A}\left(\frac{BQ}{R_0}\right)$ implies $k^{LB} > 0$ and hence $\overline{A}(Q) < \overline{A}\left(\frac{BQ}{R_0}\right)$.

Because solutions to (6) are not unique for $A \in \left(\overline{A}(Q), \overline{A}\left(\frac{BQ}{R_0}\right)\right)$, we assume that if there are multiple combinations of (k, X) that achieve the same payoff $\widehat{U}_H$, H-banks choose the one with minimum borrowing. This yields a unique solution $\left(k^{LB}, X^{LB}\right)$, where k^{LB} is given in (8) and

$$X^{LB} \equiv \underline{X}\left(k^{LB}\right) = \frac{R_0}{B - R_0}\left(A - V - Qk^{LB}\right) = \frac{R_0}{B - R_0}\left(A - \overline{A}(Q)\right). \tag{11}$$

The super script LB stands for "leveraged buy". All in all, borrowing $X^{LB} \in \left[0, \frac{R_0}{B}(A - V)\right]$ given in (11) and buying $k^{LB} \in \left[\frac{R_0}{B}\frac{A-V}{Q}, \frac{A-V}{Q}\right]$ given in (8) achieves the payoff $\widehat{U}_H$ if and only if $A \in \left[\overline{A}(Q), \overline{A}\left(\frac{BQ}{R_0}\right)\right]$. Note that this nests a boundary "no borrowing" case $\left(k^{LB}, X^{LB}\right) = \left(\frac{A-V}{Q}, 0\right)$ for $A = \overline{A}(Q)$.

Borrow and buy $Qk > A - V$. In this case, H-banks need to borrow at least

$$Qk - (A - V) \equiv \underline{\underline{X}}(k) \in (0, Qk). \tag{12}$$

With limited liability, the failure of risky investment means zero profit for any $k > \frac{A-V}{Q}$. By borrowing $X \in \left[\underline{\underline{X}}(k), Qk\right]$, a profit for H-banks is

$$\widehat{\pi}_H\left(k > \frac{A-V}{Q}, X\right) = \begin{cases} R(V+k) - BX & \text{with probability } p_H \\ 0 & \text{with probability } 1 - p_H. \end{cases}$$

Proceeding similarly as before, we denote the success profit by $\Gamma_2(k, X) \equiv R(V+k) - BX$. H-banks' problem can now be written as

$$\max_{(k,X)}\left\{p_H\Gamma_2(k, X) - \frac{\gamma}{2}\omega_H\left(\Gamma_2(k, X)\right)^2\right\} \tag{13}$$

$$\text{subject to } k > \frac{A-V}{Q} \text{ and } X \geq \underline{\underline{X}}(k).$$

An iso-payoff line $\left\{(k,X)\,|\Gamma_2(k,X) = \frac{1}{\gamma}\frac{1}{1-p_H}\right\}$ can be written as

$$X = \frac{R}{B}k - \frac{R}{B}\left(\frac{1}{\gamma R}\frac{1}{1-p_H} - V\right) = \frac{R}{B}\left(k - k^{LB}\right) \equiv X_2^{IP}(k), \tag{14}$$

where k^{LB} was defined in (8). Because $\frac{R}{B} > Q$, solutions to the problem (13) are characterized as a line segment of (14) that is above $\underline{\underline{X}}(k)$ on the (k,X)-plane. Note that $\underline{\underline{X}}(k)$ decreases in A while $X_2^{IP}(k)$ is independent of A. Also, $X_2^{IP}\left(k^{LB}\right) = 0$. Therefore, solutions to the problem (13) are greater than k^{LB} whenever they exist. Because H-banks choose to minimize borrowing when multiple (k,X) can achieve the payoff $\widehat{U}_H = \frac{1}{2\gamma}\frac{p_H}{1-p_H}$, the unique solution is given by the intersection of $\underline{\underline{X}}(k)$ and $X_2^{IP}(k)$, which is

$$k^{LB}(Q) \equiv \frac{Rk^{LB} - B(A-V)}{R-QB} = \frac{\frac{1}{\gamma}\frac{1}{1-p_H} - \{BA + (R-B)V\}}{R-QB}. \tag{15}$$

$$X^{LB}(Q) \equiv \underline{\underline{X}}\left(k^{LB}(Q)\right) = \frac{R}{R-QB}\left\{\overline{A}(Q) - A\right\}. \tag{16}$$

Comparing k^{LB} with $k^{LB}(Q)$, $k^{LB} > k^{LB}(Q) \Leftrightarrow (R-QB)\,k^{LB} > Rk^{LB} - B(A-V) \Leftrightarrow Qk^{LB} < A-V$. Therefore, k^{LB} must be positive for $k^{LB}(Q)$ to be positive. Given $R > QB$ and $k^{LB} > 0$, it is straightforward to show the following relationship:

$$\begin{aligned} \tfrac{A-V}{Q} < k^{LB} < k^{LB}(Q) \quad &\Leftrightarrow \quad A < \overline{A}(Q), \\ k^{LB}(Q) < k^{LB} < \tfrac{A-V}{Q} \quad &\Leftrightarrow \quad \overline{A}(Q) < A. \end{aligned}$$

All in all, borrowing $X^{LB}(Q)$ given in (16) and buying $k^{LB}(Q) > \frac{A-V}{Q}$ given in (15) achieves the payoff $\widehat{U}_H$ if and only if $A \in \left[V, \overline{A}(Q)\right)$.

Summary. With limited liability, H-banks can achieve the payoff $\widehat{U}_H \equiv \frac{1}{2\gamma}\frac{p_H}{1-p_H}$ by submitting $k^{LB} \in \left[\frac{R_0}{B}\frac{A-V}{Q}, \frac{A-V}{Q}\right]$ or $k^{LB}(Q) > \frac{A-V}{Q}$ under the following conditions:

$$\frac{R}{Q} > B > R_0 \qquad \text{and} \qquad A \leq \overline{A}\left(\frac{B}{R_0}Q\right), \tag{17}$$

$$A \in \left[\overline{A}(Q), \overline{A}\left(\frac{B}{R_0}Q\right)\right] \quad \Rightarrow \quad k^{LB} \in \left[\frac{R_0}{B}\frac{A-V}{Q}, \frac{A-V}{Q}\right], \tag{18}$$

$$A \in \left[V, \overline{A}(Q)\right) \quad \Rightarrow \quad k^{LB}(Q) > \frac{A-V}{Q} \tag{19}$$

In addition to (17), if $\widehat{U}_H \equiv \frac{1}{2\gamma}\frac{p_H}{1-p_H} \geq U_H^*(Q)$, then leveraged buy $\left(k^{LB}, X^{LB}\right)$ or $\left(k^{LB}(Q), X^{LB}(Q)\right)$ is indeed optimal for H-banks.

4 Ex ante investment

Before turning to the analysis of ex ante decision making, we point out that there are two types of market-clearing equilibria at the ex post trading stage.

Definition 1 (market-clearing equilibrium)

Q^*-equilibrium is a market-clearing equilibrium where H-banks submit $k_H^*(Q)$ and Q^* clears the market, i.e., $\lambda k_H^*(Q^*) + (1-\lambda) k_L^*(Q^*) = 0$.

Q^{LB}-equilibrium is a market-clearing equilibrium where H-banks submit k^{LB} or $k^{LB}(Q)$ *and Q^{LB} clears the market, i.e., $\lambda k^{LB} + (1-\lambda) k_L^*\left(Q^{LB}\right) = 0$ or $\lambda k^{LB}(Q) + (1-\lambda) k_L^*\left(Q^{LB}\right) = 0$.*

First, consider Q^*-equilibrium. If (17) holds at $Q = Q^*$, then achieving the payoff $\widehat{U}_H$ by submitting k^{LB} or $k^{LB}(Q^*)$ is feasible. In this case, the Q^*-equilibrium must satisfy

$$U_H^*(Q^*) \geq \widehat{U}_H \equiv \frac{1}{2\gamma}\frac{p_H}{1-p_H}. \tag{20}$$

If (17) does not hold at $Q = Q^*$, then (20) is not necessary for the Q^*-equilibrium.

Second, consider Q^{LB}-equilibrium. Because submitting $k_H^*\left(Q^{LB}\right)$ is always individually feasible, it must satisfy (17) at $Q = Q^{LB}$, and

$$\widehat{U}_H \equiv \frac{1}{2\gamma}\frac{p_H}{1-p_H} \geq U_H^*\left(Q^{LB}\right). \tag{21}$$

From (20) and (21), two types of equilibria may coexist if and only if

$$\widehat{U}_H \in \left[U_H^*\left(Q^{LB}\right), U_H^*\left(Q^*\right)\right] \neq \varnothing.$$

It follows that whenever the two equilibria coexist, H-banks prefer Q^*-equilibrium to Q^{LB}-equilibrium.

At the ex ante stage, banks are identical. Therefore, they solve the following problem. In a Q^*-equilibrium,

$$\max_{V\in[0,A]} \left\{\lambda U_H^*\left(Q^*\right) + (1-\lambda) U_L^*\left(Q^*\right)\right\}$$

$$\text{s.t. (20) if (17) holds at } Q^*.$$

In a Q^{LB}-equilibrium,

$$\max_{V\in[0,A]} \left\{\lambda \widehat{U}_H + (1-\lambda) U_L^*\left(Q^{LB}\right)\right\}$$

$$\text{s.t. (17) at } Q^{LB} \text{ and (21).}$$

We assume that when two types of equilibria coexist (i.e., for a set of values of V for which both problems above are well-defined), banks choose the one with the higher ex ante payoff. Because at the ex post stage H-banks always prefer a Q^*-equilibrium to a Q^{LB}-equilibrium for the same V, a sufficient condition for a Q^*-equilibrium to be chosen ex ante is that $U_L^*\left(Q^*\right) \geq U_L^*\left(Q^{LB}\right)$ for the same V.

5 Characterizing Q^*-equilibrium

In this section, we focus on a Q^*-equilibrium. A market-clearing price Q^* solves the market-clearing condition

$$\lambda k_H^*(Q) + (1-\lambda) k_L^*(Q) = 0, \tag{22}$$

where $k_L^*(Q)$ and $k_H^*(Q)$ are given in (2). Depending on whether $k_i^*(Q)$ is constrained or not for $i \in \{L, H\}$, there are four potential cases to consider:

		Buy orders	
		$\lambda\left(\frac{p_H R - Q R_0}{\gamma \omega_H R^2} - V\right)$	$\lambda \frac{A-V}{Q}$
Sell	$(1-\lambda)\left(\frac{p_L R - Q R_0}{\gamma \omega_L R^2} - V\right)$	Q_1^*-eqb	Q_3^*-eqb
orders	$-(1-\lambda) V$	Q_2^*-eqb	Q_4^*-eqb

For example, all banks submit unconstrained orders in Q_1^*-equilibrium, while all banks submit constrained orders in Q_4^*-equilibrium.

Definition 2 (robust Q^*-equilibrium) *Q_k^*-equilibrium is robust if either:*

(Type I robustness) *(17) does not hold, or*

(Type II robustness) *(17) and $U_H^*(Q_k^*) \geq \widehat{U}_H$ hold.*

In **Definition 2**, Type I robustness is the case where $\widehat{U}_H$ is not achievable at Q_k^*, while Type II robustness is the case where $\widehat{U}_H$ is achievable but no greater than $U_H^*(Q_k^*)$.

First, we ignore the robustness and derive a market-clearing price Q_k^*. We characterize conditions in terms of (V, A, R) under which each case arises (**Lemmas 1** through **4**). We define the following variables:

$$\Omega \equiv \left\{\frac{\lambda}{\omega_H} + \frac{1-\lambda}{\omega_L}\right\}^{-1}, \quad \omega \equiv \frac{\lambda \omega_L}{\lambda \omega_L + (1-\lambda)\omega_H} \in (0,1), \quad p^* \equiv \omega p_H + (1-\omega) p_L,$$

$$\widehat{V} \equiv \lambda \frac{p_H - p_L}{\gamma R \omega_H}. \tag{23}$$

Note that Ω is a harmonic mean of $\{\omega_H, \omega_L\}$. Also, Ω and ω satisfy $\frac{\omega}{\Omega} = \frac{\lambda}{\omega_H}$ and $\frac{1-\omega}{\Omega} = \frac{1-\lambda}{\omega_L}$.

Lemma 1 *Equilibrium with* $Q_1^* = \frac{R}{R_0}(p^* - \gamma\Omega RV)$ *arises for* (V, A, R) *such that* $V \geq \widehat{V}$ *and*

$$\frac{p_H - Q_1^* \frac{R_0}{R}}{\gamma R \omega_H} Q_1^* + V(1 - Q_1^*) \leq A. \tag{24}$$

Proof. The market-clearing condition (22) becomes $\lambda \frac{p_H - Q\frac{R_0}{R}}{\gamma R \omega_H} + (1-\lambda)\frac{p_L - Q\frac{R_0}{R}}{\gamma R \omega_L} = V$. Solving this for Q yields the expression of Q_1^*. For sellers not to be constrained, we need $p_L R \geq Q_1^* R_0$, which is equivalent to $V \geq \widehat{V}$. For buyers not to be constrained, we need $\frac{p_H - Q_1^*\frac{R_0}{R}}{\gamma R \omega_H} - V \leq \frac{A-V}{Q_1^*}$, which is equivalent to (24). ■

Lemma 2 *Equilibrium with* $Q_2^* = \frac{R}{R_0}\left(p_H - \gamma\omega_H \frac{RV}{\lambda}\right)$ *arises for* (V, A, R) *such that* $V < \widehat{V}$ *and*

$$\frac{p_H - Q_2^* \frac{R_0}{R}}{\gamma R \omega_H} Q_2^* + V(1 - Q_2^*) \leq A. \tag{25}$$

Proof. The market-clearing condition (22) becomes $\lambda \frac{p_H - Q\frac{R_0}{R}}{\gamma R \omega_H} = V$. Solving this for Q yields the expression of Q_2^*. For sellers to be constrained, we need $p_L R < Q_2^* R_0$, which is equivalent to $V < \widehat{V}$. For buyers not to be constrained, we need $\frac{p_H - Q_2^*\frac{R_0}{R}}{\gamma R \omega_H} - V \leq \frac{A-V}{Q_2^*}$, which is equivalent to (25). ■

Lemma 3 *Equilibrium with* $Q_3^* \in \left(0, \frac{p_L R}{R_0}\right)$*, a unique solution to* $\lambda\frac{A-V}{Q} + (1-\lambda)\left(\frac{p_L - Q\frac{R_0}{R}}{\gamma R \omega_L}\right) = (1-\lambda)V$*, arises for* (V, A, R) *such that*

$$A < \min\left\{\left(\frac{1-\lambda}{\lambda}\frac{p_L R}{R_0} + 1\right)V, \left(\frac{p_H - Q_3^*\frac{R_0}{R}}{\gamma R \omega_H} - V\right)Q_3^* + V\right\}.$$

Proof. For sellers not to be constrained, we need $Q_3^* < \frac{p_L R}{R_0}$. The market-clearing condition (22) becomes $\lambda\frac{A-V}{Q} + (1-\lambda)\left(\frac{p_L - Q\frac{R_0}{R}}{\gamma R \omega_L}\right) = (1-\lambda)V$. The left hand side is decreasing in Q, and it increases without bound as $Q \to 0$, while it decreases to $\lambda(A-V)\frac{R_0}{p_L R}$ as $Q \to \frac{p_L R}{R_0}$. Therefore, $\lambda(A-V)\frac{R_0}{p_L R} < (1-\lambda)V$ is necessary and sufficient for the existence of the unique solution $Q_3^* \in \left(0, \frac{p_L R}{R_0}\right)$. This condition is equivalent to $A < \left(\frac{1-\lambda}{\lambda}\frac{p_L R}{R_0} + 1\right)V$.

For buyers to be constrained, we need $\frac{p_H - Q_3^* \frac{R_0}{R}}{\gamma R \omega_H} - V > \frac{A-V}{Q_3^*}$, which is equivalent to $A < \left(\frac{p_H - Q_3^* \frac{R_0}{R}}{\gamma R \omega_H} - V \right) Q_3^* + V$. ■

Lemma 4 *Equilibrium with* $Q_4^* = \frac{1-\lambda}{\lambda} \frac{A-V}{V}$ *arises for* (V, A, R) *such that* $V < \widehat{V}$ *and*

$$\left(\frac{1-\lambda}{\lambda} \frac{p_L R}{R_0} + 1 \right) V < A < \left\{ \frac{1-\lambda}{\lambda} \frac{R}{R_0} \left(p_H - \gamma \omega_H \frac{RV}{\lambda} \right) + 1 \right\} V.$$

Proof. The market-clearing condition (22) becomes $\lambda \frac{A-V}{Q} = (1-\lambda) V$. Solving this for Q yields the expression of Q_4^*. For sellers to be constrained, we need $p_L R < Q_4^* R_0$, which is equivalent to $\left(\frac{1-\lambda}{\lambda} \frac{p_L R}{R_0} + 1 \right) V < A$. For buyers to be constrained, we need $\frac{p_H - Q_4^* \frac{R_0}{R}}{\gamma R \omega_H} - V > \frac{A-V}{Q_4^*}$, which is equivalent to $A < \left\{ \frac{1-\lambda}{\lambda} \frac{R}{R_0} \left(p_H - \gamma \omega_H \frac{RV}{\lambda} \right) + 1 \right\} V$. For any $V > 0$, the specified range for A is nonempty if and only if $V < \widehat{V}$. ■

To characterize the robustness condition, recall from (*4*) that the payoff of H-banks in Q_k^*-equilibrium, $k \in \{1, 2\}$, is

$$U_H^* \left(Q_k^* \right) = \left\{ A + \left(Q_k^* - 1 \right) V \right\} R_0 + \frac{1}{2\gamma} \frac{p_H}{1 - p_H} \left(\frac{p_H R - Q_k^* R_0}{p_H R} \right)^2.$$

Then $U_H^* \left(Q_k^* \right) \geq \widehat{U}_H$ for $k \in \{1, 2\}$ is

$$\left\{ A + \left(Q_k^* - 1 \right) V \right\} R_0 + \frac{1}{2\gamma} \frac{p_H}{1 - p_H} \left(\frac{p_H R - Q_k^* R_0}{p_H R} \right)^2 \geq \frac{1}{2\gamma} \frac{p_H}{1 - p_H}$$

which is equivalent to

$$\frac{p_H - \frac{1}{2} Q_k^* \frac{R_0}{R}}{\gamma R \omega_H} Q_k^* + V \left(1 - Q_k^* \right) \leq A, \qquad k \in \{1, 2\}. \tag{26}$$

The left hand side of (26) is independent of A for $k \in \{1, 2\}$.

Proposition 1

(a) *The condition (26) implies (24) for* $k = 1$ *and (25) for* $k = 2$.

(b) $U_H^*(Q_k^*) < \widehat{U}_H$ *for* $k \in \{3, 4\}$.

Proof.

(a) For $Q_k^* > 0$, the left-hand side of (26) is greater than the left-hand side of (24) and that of (25).

(b) From (4), the payoff of H-banks submitting $\frac{A-V}{Q}$ is

$$U_H\left(\frac{A-V}{Q}; Q\right) = \left[\{A + (Q-1)V\}\frac{p_H R}{Q}\right] - \frac{\gamma}{2}\frac{1-p_H}{p_H}\left[\{A + (Q-1)V\}\frac{p_H R}{Q}\right]^2.$$

This takes the maximum value $\frac{1}{2\gamma}\frac{p_H}{1-p_H}$ when $\frac{p_H R}{Q}\{A + (Q-1)V\} = \frac{1}{\gamma}\frac{p_H}{1-p_H}$. Otherwise, $U_H\left(\frac{A-V}{Q}; Q\right) < \frac{1}{2\gamma}\frac{p_H}{1-p_H}$. Because $\frac{p_H R}{Q}\{A + (Q-1)V\} = \frac{1}{\gamma}\frac{p_H}{1-p_H} \Leftrightarrow R(A-V) = Q\left(\frac{1}{\gamma}\frac{1}{1-p_H} - RV\right)$, it suffices to show that $R(A-V) \neq Q_k^*\left(\frac{1}{\gamma}\frac{1}{1-p_H} - RV\right)$ for $k \in \{3, 4\}$.

For Q_3^*, recall from **Lemma 3** that $R(A-V) < \min\left\{\frac{1-\lambda}{\lambda}\frac{p_L R}{R_0}RV, \left(\frac{Rp_H - Q_3^* R_0}{\gamma R \omega_H} - RV\right)Q_3^*\right\}$. Because $\left(\frac{Rp_H - Q_3^* R_0}{\gamma R \omega_H} - RV\right)Q_3^* < \left(\frac{1}{\gamma}\frac{1}{1-p_H} - RV\right)Q_3^*$, it follows $R(A-V) < Q_3^*\left(\frac{1}{\gamma}\frac{1}{1-p_H} - RV\right)$.

For Q_4^*, use $Q_4^* = \frac{1-\lambda}{\lambda}\frac{A-V}{V}$ to rewrite $R(A-V) = Q_4^*\left(\frac{1}{\gamma}\frac{1}{1-p_H} - RV\right)$ to $RV = \frac{\lambda}{\gamma}\frac{1}{1-p_H}$. This cannot be satisfied because, from **Lemma 4**, $RV < \frac{\lambda}{\gamma\omega_H}(p_H - p_L) = \frac{\lambda}{\gamma}\frac{1}{1-p_H}\frac{p_H - p_L}{p_H} < \frac{\lambda}{\gamma}\frac{1}{1-p_H}$. ■

By **Proposition 1**, the condition $U_H^*(Q_k^*) \geq \widehat{U}_H$ (i.e., Type II robustness) needs to be checked only for $k \in \{1, 2\}$. For $k \in \{3, 4\}$, Q_k^*-equilibrium characterized in **Lemma 3** or **4** is robust if and only if (17) does not hold (i.e., Type I-robustness is the only possibility).

To characterize Type II robust Q_1^*- and Q_2^*-equilibrium, we define the following functions and variables.

$$G_1(V) \equiv a_1 V^2 + b_1 V + c_1, \qquad \text{where}$$

$$a_1 \equiv \frac{R}{R_0}\frac{\gamma\Omega R}{2}\left(2 - \frac{\Omega}{\omega_H}\right),\ b_1 \equiv -\frac{R}{R_0}\left(\frac{\Omega}{\omega_H}p_H + \left(1 - \frac{\Omega}{\omega_H}\right)p^* - \frac{R_0}{R}\right),\ c_1 \equiv \frac{1}{R_0}\frac{1}{2\gamma}\frac{p^*(2p_H - p^*)}{\omega_H}.$$

$$G_2(V) \equiv a_2 V^2 + b_2 V + c_2, \qquad \text{where}$$

$$a_2 \equiv \frac{R}{R_0}\frac{\gamma\omega_H R}{\lambda}\frac{2\lambda-1}{2\lambda}, \quad b_2 \equiv -\frac{R}{R_0}\left(p_H - \frac{R_0}{R}\right), \quad c_2 \equiv \frac{1}{R_0}\frac{1}{2\gamma}\frac{p_H}{1-p_H}.$$

Note that $R_0 c_2 = \frac{1}{2\gamma}\frac{p_H}{1-p_H} \equiv \widehat{U}_H$.

Lemma 5 (G_1 and G_2)

(a) *The condition (26) is equivalent to* $G_k(V) \leq A$. *Also,* $G_1\left(\widehat{V}\right) = G_2\left(\widehat{V}\right)$.

(b) $b_1 < b_2$ *and* $0 < c_1 < c_2$.

(c) $a_2 < 0 < a_1 \Leftrightarrow \lambda \in \left(\frac{1}{2}\frac{\omega_L - 2\omega_H}{\omega_L - \omega_H}, \frac{1}{2}\right)$.

(d) *For* $\lambda < \frac{1}{2}$, $G_1(V)$ *is increasing in* $V \geq 0$ *if and only if* $-\frac{b_1}{2a_1} \leq 0$.

Otherwise, $G_1(V)$ *takes the minimum value at* $V = -\frac{b_1}{2a_1} > 0$.

(e) *For* $\lambda > \frac{1}{2}\frac{\omega_L - 2\omega_H}{\omega_L - \omega_H}$, $G_2(V)$ *is decreasing in* $V \geq 0$ *for any given* $R \geq \frac{R_0}{p_H}$ *while it takes the maximum value at* $V = -\frac{b_2}{2a_2} > 0$ *for any given* $R < \frac{R_0}{p_H}$.

Proof.

(a) Substituting the expression of Q_k^* into the left-hand side of (26) yields $G_k(V) \leq A$. From **Lemmas 1** and **2**, $Q_1^*\left(\widehat{V}\right) = Q_2^*\left(\widehat{V}\right) = \frac{p_L R}{R_0}$, which implies $G_1\left(\widehat{V}\right) = G_2\left(\widehat{V}\right)$.

(b) First, $\frac{\Omega}{\omega_H}p_H + \left(1 - \frac{\Omega}{\omega_H}\right)p^* = p_H + \left(\frac{\Omega}{\omega_H} - 1\right)(p_H - p^*)$. Then $\frac{\Omega}{\omega_H} = \frac{\omega}{\lambda} = \frac{\omega_L}{\lambda\omega_L + (1-\lambda)\omega_H} \in \left(1, \frac{\omega_L}{\omega_H}\right)$ implies that $b_1 = b_2 - \frac{R}{R_0}\left(\frac{\Omega}{\omega_H} - 1\right)(p_H - p^*) < b_2$. Second, $c_1 = c_2\frac{p^*}{p_H}\left(2 - \frac{p^*}{p_H}\right) > 0$, where $\frac{p^*}{p_H}\left(2 - \frac{p^*}{p_H}\right) \leq 1$ with the equality holding if and only if $\frac{p^*}{p_H} = 1 \Leftrightarrow \lambda = 1$.

(c) First, $a_2 < 0 \Leftrightarrow \lambda < \frac{1}{2}$ is obvious. Second, $0 < a_1 \Leftrightarrow \Omega < 2\omega_H \Leftrightarrow \frac{1}{\frac{\lambda}{\omega_H} + \frac{1-\lambda}{\omega_L}} < 2\omega_H \Leftrightarrow 1 < 2\lambda + 2(1-\lambda)\frac{\omega_H}{\omega_L} \Leftrightarrow 1 - 2\frac{\omega_H}{\omega_L} < 2\lambda\left(1 - \frac{\omega_H}{\omega_L}\right) \Leftrightarrow \frac{1}{2}\frac{\omega_L - 2\omega_H}{\omega_L - \omega_H} < \lambda$.

(d,e) This follows from the property of quadratic equations G_1 and G_2. ■

Given **Lemma 5**, we define the following variables:

$$\underline{V}_1 \equiv -\frac{b_1}{2a_1} = \frac{p_H + \left(\frac{\omega}{\lambda} - 1\right)(p_H - p^*) - \frac{R_0}{R}}{\gamma R\Omega\left(2 - \frac{\omega}{\lambda}\right)},$$

$$\overline{V}_2 \equiv -\frac{b_2}{2a_2} = -\frac{\lambda^2}{1-2\lambda}\frac{p_H - \frac{R_0}{R}}{\gamma R\omega_H},$$

$$\underline{A}_1 \equiv G_1\left(\underline{V}_1\right) = c_1 - \frac{b_1^2}{4a_1},$$

$$\underline{A}_2 \equiv G_2\left(\widehat{V}\right) = G_1\left(\widehat{V}\right).$$

Importantly, $a_2 < 0 < a_1$ implies

$$\underline{V}_1 = \arg\min_V G_1(V), \qquad \underline{A}_1 = \min_V G_1(V), \qquad \overline{V}_2 = \arg\max_V G_2(V).$$

Recall that Q_1^*-equilibrium requires $V \geq \widehat{V}$ while Q_2^*-equilibrium requires $V < \widehat{V}$. Because $\widehat{V}$, $\underline{V}_1$, and $\overline{V}_2$ all depend on R, we need to consider various threshold values of R to consider the ex ante optimal choice of V.

We denote the ex ante optimal choice of V by V^*. For $k \in \{1, 2\}$, let V_k^* be the ex ante optimal choice of V in Q_k^*-equilibrium that ignores the condition (26).

Definition 3 *Q_k^*-equilibrium, $k \in \{1, 2\}$, is unconstrained if the constraint (26) is not binding, i.e., $V^* = V_k^*$ and $A \geq G_k(V_k^*)$. It is constrained if the constraint (26) is binding, i.e., $G_k(V_k^*) > A \geq G_k(V^*)$.*

We first characterize an unconstrained equilibrium.

Proposition 2 (unconstrained equilibrium)

An unconstrained Q_k^*-equilibrium, $k \in \{1, 2\}$, with positive V^* is robust if $A \geq \underline{A}^* \equiv \frac{p_H - \frac{1}{2}\frac{R_0}{R}}{\gamma R \omega_H}$ and $R > \frac{R_0}{p_H}$. In particular, Q_1^*-equilibrium with $V_1^* = \frac{p^* - \frac{R_0}{R}}{\gamma R \Omega} \geq \widehat{V}$ and $Q_1^* = 1$ prevails for $R \geq \frac{R_0}{p_L}$, while Q_2^*-equilibrium with $V_2^* = \lambda \frac{p_H - \frac{R_0}{R}}{\gamma R \omega_H} \in \left(0, \widehat{V}\right)$ and $Q_2^* = 1$ prevails for $R \in \left(\frac{R_0}{p_H}, \frac{R_0}{p_L}\right)$.

Proof. With $Q_k^* = 1$, the constraint (26) becomes $A \geq \underline{A}^* \equiv \frac{p_H - \frac{1}{2}\frac{R_0}{R}}{\gamma R \omega_H}$. Hence it suffices to show that choosing V that makes $Q_k^* = 1$ is ex ante optimal for $k \in \{1, 2\}$.

For Q_1^*-equilibrium, the ex ante objective is

$$\begin{aligned} U_1^* &\equiv \lambda U_H^*(Q_1^*) + (1-\lambda) U_L^*(Q_1^*) \\ &= \{A + (Q_1^* - 1) V\} R_0 + \frac{1}{2\gamma}\left[\begin{array}{c} \lambda \frac{p_H}{1-p_H}\left(\frac{p_H R - Q_1^* R_0}{p_H R}\right)^2 \\ +(1-\lambda)\frac{p_L}{1-p_L}\left(\frac{p_L R - Q_1^* R_0}{p_L R}\right)^2 \end{array}\right]. \end{aligned}$$

To take the derivative of U_1^* with respect to V, note that $\frac{\partial Q_1^*}{\partial V} = -\frac{\gamma R^2 \Omega}{R_0} < 0$ and $\frac{\partial}{\partial V}\left(\frac{p_i R - Q_1^* R_0}{p_i R}\right)^2 = 2\gamma\Omega \frac{p_i R - Q_1^* R_0}{p_i^2}$ for $i \in \{H, L\}$. Therefore,

$$\begin{aligned}
\frac{\partial U_1^*}{\partial V} &= \left\{Q_1^* - 1 - \frac{\gamma R^2 \Omega}{R_0} V\right\} R_0 + \Omega \left[\begin{array}{c} \lambda \frac{p_H}{1-p_H} \frac{p_H R - Q_1^* R_0}{p_H^2} \\ + (1-\lambda) \frac{p_L}{1-p_L} \frac{p_L R - Q_1^* R_0}{p_L^2} \end{array}\right] \\
&= (Q_1^* - 1) R_0 + \Omega \left[\lambda \frac{p_H R - Q_1^* R_0}{\omega_H} + (1-\lambda) \frac{p_L R - Q_1^* R_0}{\omega_L} - \gamma R^2 V\right] \\
&= (Q_1^* - 1) R_0.
\end{aligned}$$

The last equality holds from the market-clearing condition. Because $\frac{\partial Q_1^*}{\partial V} < 0$, there is a unique optimal V that solves $Q_1^*(V) = 1$. This yields $V_1^* = \frac{p^* - \frac{R_0}{R}}{\gamma R \Omega}$, which is positive for any $R > \frac{R_0}{p^*} > \frac{R_0}{p_H}$.

The reasoning for Q_2^*-equilibrium is similar. From (*4*), the payoff for sellers is $\{A + (Q_2^* - 1) V\} R_0$. The ex ante objective is

$$\begin{aligned}
U_2^* &\equiv \lambda U_H^*(Q_2^*) + (1-\lambda)\{A + (Q_2^* - 1) V\} R_0 \\
&= \{A + (Q_2^* - 1) V\} R_0 + \frac{\lambda}{2\gamma} \frac{p_H}{1-p_H} \left(\frac{p_H R - Q_2^* R_0}{p_H R}\right)^2.
\end{aligned}$$

To take the derivative of U_2^* with respect to V, note that $\frac{\partial Q_2^*}{\partial V} = -\frac{\gamma R^2}{R_0} \frac{\omega_H}{\lambda} < 0$ and $\frac{\partial}{\partial V}\left(\frac{p_H R - Q_2^* R_0}{p_H R}\right)^2 = \frac{2\gamma}{\lambda} \frac{1-p_H}{p_H} (p_H R - Q_2^* R_0)$. Therefore,

$$\begin{aligned}
\frac{\partial U_2^*}{\partial V} &= \left\{Q_2^* - 1 - \frac{\gamma R^2}{R_0} \frac{\omega_H}{\lambda} V\right\} R_0 + (p_H R - Q_2^* R_0) \\
&= (Q_2^* - 1) R_0 + p_H R - Q_2^* R_0 - \gamma R^2 \frac{\omega_H}{\lambda} V \\
&= (Q_2^* - 1) R_0.
\end{aligned}$$

The last equality holds from the market-clearing condition. Because $\frac{\partial Q_2^*}{\partial V} < 0$, there is a unique optimal V that solves $Q_2^*(V) = 1$. This yields $V_2^* = \lambda \frac{p_H - \frac{R_0}{R}}{\gamma R \omega_H}$, which is positive for any $R > \frac{R_0}{p_H}$.

Finally, it is straightforward to show that $V_1^* < V_2^* < \widehat{V} \Leftrightarrow R < \frac{R_0}{p_L}$ and $V_1^* > V_2^* > \widehat{V} \Leftrightarrow R > \frac{R_0}{p_L}$. ■

From the proof of **Proposition 2**, $G_1(V_1^*) = G_2(V_2^*) = \frac{p_H - \frac{1}{2}\frac{R_0}{R}}{\gamma R \omega_H} \equiv \underline{A}^*$. Because $\underline{A}^*$ is maximized at $R = \frac{R_0}{p_H}$ with the maximum value c_2, $\underline{A}^*$ is decreasing in R for $R > \frac{R_0}{p_H}$. Also, $\underline{A}^* = G_2(V_2^*)$ and $\underline{A}_2 \equiv G_2\left(\widehat{V}\right)$ intersect at $R = \frac{R_0}{p_L} \Leftrightarrow V_2^* = \widehat{V}$. Because $G_2(V)$ is decreasing in V for any $R > \frac{R_0}{p_H}$, $R \lesseqgtr \frac{R_0}{p_L} \Leftrightarrow V_2^* \lesseqgtr \widehat{V} \Leftrightarrow \underline{A}^* \gtreqless \underline{A}_2$. Also, because $\overline{V}_2 = -\frac{\lambda^2}{1-2\lambda}\frac{p_H - \frac{R_0}{R}}{\gamma\omega_H R} = -\frac{\lambda}{1-2\lambda}V_2^*$, $\overline{V}_2$ and V_2^* have opposite signs and $\left|\overline{V}_2\right| < |V_2^*|$ for $\lambda < \frac{1}{2}$. With these properties, we characterize the relationship between $\widehat{V}$ and $\underline{V}_1$, $\overline{V}_2$. It is convenient to define the following two variables.

$$\widehat{p} \equiv p_H - \left(1 + \omega - \frac{\omega}{\lambda}\right)(p_H - p_L) \qquad \text{and} \qquad \widetilde{p} \equiv p_H + \frac{1-2\lambda}{2\lambda}(p_H - p_L).$$

Lemma 6 ($\widehat{V}$ and $\underline{V}_1$, $\overline{V}_2$)

(a) $\widehat{V} \gtreqless \underline{V}_1 \Leftrightarrow R \lesseqgtr \frac{R_0}{\widehat{p}}$.

(b) $\widehat{p} > p^* \Leftrightarrow \lambda < \frac{1}{2}$.

(c) $\widehat{p} \lesseqgtr p_H \Leftrightarrow \lambda \gtreqless \frac{1}{2}\frac{\omega_L - \omega_H}{\omega_L - \frac{1}{2}\omega_H}$.

(d) *Given* $\lambda < \frac{1}{2}$ *and* $R < \frac{R_0}{p_H}$, $c_2 > \underline{A}_2 \Leftrightarrow 2\overline{V}_2 < \widehat{V} \Leftrightarrow \frac{R_0}{\widetilde{p}} < R$.

Proof.

(a) $\widehat{V} < \underline{V}_1 \Leftrightarrow \lambda\frac{p_H - p_L}{\gamma R \omega_H} < \frac{p_H + \left(\frac{\omega}{\lambda} - 1\right)(p_H - p^*) - \frac{R_0}{R}}{\gamma R \Omega\left(2 - \frac{\omega}{\lambda}\right)} \Leftrightarrow \frac{R_0}{R} < p_H + \left(\frac{\omega}{\lambda} - 1\right)(1-\omega)(p_H - p_L) - \frac{\lambda\Omega\left(2 - \frac{\omega}{\lambda}\right)}{\omega_H}(p_H - p_L) = p_H + \left\{\left(\frac{\omega}{\lambda} - 1\right)(1-\omega) - \omega\left(2 - \frac{\omega}{\lambda}\right)\right\}(p_H - p_L) = \widehat{p}$.

(b) From $p^* = p_H - (1-\omega)(p_H - p_L)$, $\widehat{p} - p^* = \left\{(1-\omega) - \left(1 + \omega - \frac{\omega}{\lambda}\right)\right\}(p_H - p_L) = \left(\frac{1}{\lambda} - 2\right)\omega(p_H - p_L)$. Therefore, $\widehat{p} > p^* \Leftrightarrow \lambda < \frac{1}{2}$.

(c) $1 + \omega - \frac{\omega}{\lambda} = \frac{\lambda\frac{2\omega_L - \omega_H}{\omega_L - \omega_H} - 1}{\lambda + \frac{\omega_H}{\omega_L - \omega_H}} > 0 \Leftrightarrow \lambda > \frac{1}{2}\frac{\omega_L - \omega_H}{\omega_L - \frac{1}{2}\omega_H}$.

(d) $R < \frac{R_0}{p_H}$ implies $\overline{V}_2 > 0$. Therefore, $c_2 = G_2(0) > \underline{A}_2 = G_2\left(\overline{V}_2\right) \Leftrightarrow 2\overline{V}_2 < \widehat{V}$. The last inequality can be written as $\frac{2\lambda}{1-2\lambda}\left(\frac{R_0}{R} - p_H\right) < p_H - p_L$, which is equivalent to $\frac{R_0}{R} < p_H + \frac{1-2\lambda}{2\lambda}(p_H - p_L) = \widetilde{p}$. ■

Given **Lemma 6**, we assume the following for λ.

Assumption 2 $\quad \frac{1}{2}\frac{\omega_L-\omega_H}{\omega_L-\frac{1}{2}\omega_H} < \lambda < \frac{1}{2}$.

Assumption 2 states that H-banks are not the majority and they are not too different from L-banks. Because $\frac{1}{2}\frac{\omega_L-\omega_H}{\omega_L-\frac{1}{2}\omega_H} > \frac{1}{2}\frac{\omega_L-2\omega_H}{\omega_L-\omega_H}$, **Assumption 2** is sufficient for $a_2 < 0 < a_1$ and $p^* < \widehat{p} < p_H < \widetilde{p}$.[1]

Now we are ready to characterize the constrained Q_k^*-equilibrium, $k \in \{1, 2\}$. For a decreasing part of $G_k(V)$, we define the inverse function $G_k^{-1}(A)$. The relevant domain for $G_1(V)$ is $V \in \left[\widehat{V}, \underline{V}_1\right]$ for $R > \frac{R_0}{\widehat{p}}$, and that for $G_2(V)$ is $V \geq \max\left\{0, \overline{V}_2\right\}$ for $R \leq \frac{R_0}{p_L}$.

Proposition 3 (constrained equilibrium)

Assume $\lambda \in \left(\frac{1}{2}\frac{\omega_L-\omega_H}{\omega_L-\frac{1}{2}\omega_H}, \frac{1}{2}\right)$ *so that* $a_2 < 0 < a_1$ *and* $\widehat{p} < p_H < \widetilde{p}$.

(a) *For* $R > \frac{R_0}{p_L}$, $\widehat{V} < V_2^* < V_1^* < \underline{V}_1$ *holds. A constrained* Q_1^**-equilibrium with positive* V^* *is robust if* $A \in [\underline{A}_1, \underline{A}^*)$. *It satisfies* $V^* = G_1^{-1}(A) \in (V_1^*, \underline{V}_1]$ *and* $Q_1^*(V^*) < 1$.

(b) *For* $\frac{R_0}{\widehat{p}} < R \leq \frac{R_0}{p_L}$, $\max\{0, V_1^*\} \leq V_2^* \leq \widehat{V} < \underline{V}_1$ *holds. A constrained* Q_k^**-equilibrium,* $k \in \{1, 2\}$, *with positive* V^* *is robust if* $A \in [\underline{A}_1, \underline{A}^*)$. *It satisfies*

$$V^* = G_2^{-1}(A) \in \left(V_2^*, \widehat{V}\right] \text{ and } Q_2^*(V^*) < 1 \quad \text{for } A \in [\underline{A}_2, \underline{A}^*),$$
$$V^* = G_1^{-1}(A) \in \left(\widehat{V}, \underline{V}_1\right) \text{ and } Q_1^*(V^*) < 1 \quad \text{for } A \in [\underline{A}_1, \underline{A}_2).$$

(c) *For* $\frac{R_0}{p_H} < R \leq \frac{R_0}{\widehat{p}}$, $V_1^* < 0 < V_2^* < \underline{V}_1 \leq \widehat{V}$ *holds. A constrained* Q_2^**-equilibrium with positive* V^* *is robust if* $A \in [\underline{A}_2, \underline{A}^*)$. *It satisfies* $V^* = G_2^{-1}(A) \in \left(V_2^*, \widehat{V}\right]$ *and* $Q_2^*(V^*) < 1$.

(d) *For* $\frac{R_0}{\widetilde{p}} < R \leq \frac{R_0}{p_H}$, $V_1^* < V_2^* \leq 0 \leq \overline{V}_2 < \widehat{V} < 2\overline{V}_2$ and $\underline{V}_1 < \widehat{V}$ *holds. A constrained* Q_2^**-equilibrium with positive* V^* *is robust if* $A \in [\underline{A}_2, c_2)$. *It satisfies* $V^* = G_2^{-1}(A) \in \left(G_2^{-1}(c_2), \widehat{V}\right]$ *and* $Q_2^*(V^*) < 1$. *For* $A \geq c_2$, *a robust constrained equilibrium exists but has* $V^* = 0$.

[1] Alternatively, assuming $\lambda \in \left(\frac{1}{2}\frac{\omega_L-2\omega_H}{\omega_L-\omega_H}, \frac{1}{2}\frac{\omega_L-\omega_H}{\omega_L-\frac{1}{2}\omega_H}\right)$ implies $a_2 < 0 < a_1$ and $p^* < p_H < \widehat{p}$. Proposition 3 needs only a minor adjustment.

(e) *For* $R \leq \frac{R_0}{\widetilde{p}}$, $V_1^* < V_2^* < 0 < \widehat{V} \leq 2\overline{V}_2$ *holds. A robust constrained* Q_k^**-equilibrium,* $k \in \{1,2\}$, *with positive* V^* *does not exist. For* $A \geq c_2$, *a robust constrained equilibrium exists but has* $V^* = 0$.

(f) *In any constrained* Q_k^**-equilibrium,* $k \in \{1,2\}$, *with positive* V^*, V^* *decreases in* A *and* $Q_k^*(V^*) < 1$ *increases in* A.

(g) *For case (a),* $Q_1^*(V^*) > 0$ *and* $V^* < A$ *for all* $A \in [\underline{A}_1, \underline{A}^*)$ *if and only if* $R \in \left(\frac{R_0}{p_L}, \frac{R_0}{(1-\omega)\left(\frac{\omega_L}{\omega_H}p_H - p_L\right)}\right)$. *For* $R \geq \frac{R_0}{(1-\omega)\left(\frac{\omega_L}{\omega_H}p_H - p_L\right)}$, *a constrained* Q_1^**-equilibrium with* $V^* = G_1^{-1}(A) \in \left(V_1^*, \frac{p^*}{\gamma\Omega R}\right)$ *and* $Q_1^* \in (0,1)$ *exists if and only if* $A \in \left(G_1\left(\frac{p^*}{\gamma\Omega R}\right), \underline{A}^*\right)$.

Proof.

(a)-(f) These follow from properties of $\widehat{V}$, V_1^*, V_2^*, $\underline{V}_1$, $\overline{V}_2$, $G_1(V)$, $G_2(V)$, $Q_1^*(V)$ and $Q_2^*(V)$.

(g) $\frac{p^*}{\gamma\Omega R}$ is a unique solution to $Q_1^*(V) = 0$ and also a smaller solution to $V = G_1(V)$. It is straightforward to show that $\frac{p^*}{\gamma\Omega R} \leq \underline{V}_1 \Leftrightarrow R \geq \frac{R_0}{(1-\omega)\left(\frac{\omega_L}{\omega_H}p_H - p_L\right)}$ and that $(1-\omega)\left(\frac{\omega_L}{\omega_H}p_H - p_L\right) < p_L$ under **Assumption 2**. ■

Lemma 7 $(V^* < A)$

(a) $V_1^* < \underline{A}^* \Leftarrow \frac{1}{2}\frac{\omega_L - 2\omega_H}{\omega_L - \omega_H} \leq \lambda$.

(b) $\underline{V}_1 < \underline{A}_1$ for $R \in \left(\frac{R_0}{\widehat{p}}, \frac{R_0}{(1-\omega)\left(\frac{\omega_L}{\omega_H}p_H - p_L\right)}\right)$. $G_1^{-1}(A) < A \Leftrightarrow A > G_1\left(\frac{p^*}{\gamma R\Omega}\right)$ for $R \geq \frac{R_0}{(1-\omega)\left(\frac{\omega_L}{\omega_H}p_H - p_L\right)}$.

(c) $\widehat{V} < \underline{A}_2 \Leftarrow \frac{1}{2} < p_H$.

Proof.

(a) $V_1^* \leq \underline{A}^* \Leftrightarrow \frac{p^* - \frac{R_0}{R}}{\Omega} \leq \frac{p_H - \frac{1}{2}\frac{R_0}{R}}{\omega_H} \Leftrightarrow \frac{\omega_H}{\Omega}\left(p^* - \frac{R_0}{R}\right) \leq p_H - \frac{1}{2}\frac{R_0}{R} \Leftrightarrow \left(\frac{1}{2} - \frac{\lambda}{\omega}\right)\frac{R_0}{R} \leq p_H - \frac{\lambda}{\omega}p^*$. The right-hand side is positive because $p_H - \frac{\lambda}{\omega}p^* = (1-\lambda)p_H - \frac{\lambda}{\omega}(1-\omega)p_L = (1-\lambda)p_H - \lambda\frac{(1-\lambda)\omega_H}{\lambda\omega_L}p_L = (1-\lambda)\left(p_H - \frac{\omega_H}{\omega_L}p_L\right)$. The left-hand side is non-positive if and only if $\frac{1}{2} \leq \frac{\lambda}{\omega} = \frac{\lambda\omega_L + (1-\lambda)\omega_H}{\omega_L} = \lambda\frac{\omega_L - \omega_H}{\omega_L} + \frac{\omega_H}{\omega_L} \Leftrightarrow \lambda \geq \frac{\frac{1}{2} - \frac{\omega_H}{\omega_L}}{\frac{\omega_L - \omega_H}{\omega_L}} = \frac{1}{2}\frac{\omega_L - 2\omega_H}{\omega_L - \omega_H}$.

(b) This follows from the proof of **Proposition 3(g)**.

(c) $\widehat{V} < \underline{A}_2 \Leftrightarrow 1 < a_2\widehat{V} + b_2 + \frac{c_2}{\widehat{V}} = \frac{R}{R_0}\left\{\frac{2\lambda-1}{2\lambda}(p_H - p_L) - \left(p_H - \frac{R_0}{R}\right) + \frac{1}{2\lambda}\frac{\frac{p_H}{1-p_H}}{p_H-p_L}\right\}$. This is equivalent to $(p_H - p_L)^2 < \frac{p_H}{1-p_H}$. The left hand side is no greater than one. The right-hand side is greater than one if $\frac{1}{2} < p_H$. ■

Lemma 7 implies that robust Q_k^*-equilibrium with $V^*Q_k^* \in (0, V^*]$, $k \in \{1,2\}$, constrained or not, satisfies $V^* < A$ under **Assumption 1** and **Assumption 2** .

Lemma 8 *For any $A \geq \underline{A}^* \equiv \frac{p_H - \frac{1}{2}\frac{R_0}{R}}{\gamma R \omega_H}$, an unconstrained Q_k^*-equilibrium, $k \in \{1,2\}$, satisfies $\overline{A}(Q_k^*) = \frac{1}{\gamma R}\frac{1}{1-p_H}$ and $\overline{A}\left(\frac{B}{R_0}Q_k^*\right) = \frac{B}{R_0}\frac{1}{\gamma R}\frac{1}{1-p_H} + \left(1 - \frac{B}{R_0}\right)V_k^*$ such that $\underline{A}^* < \overline{A}(Q_k^*) < \overline{A}\left(\frac{B}{R_0}Q_k^*\right)$.*

Proof. Because $Q_k^* = 1$, $\overline{A}(Q_k^*) = \frac{1}{\gamma R}\frac{1}{1-p_H} > \underline{A}^*$ trivially holds. Also, $\overline{A}\left(\frac{B}{R_0}Q_k^*\right) = V_k^* + \frac{B}{R_0}k^{LB}$, where $k^{LB} = \frac{1}{\gamma R}\frac{1}{1-p_H} - V_k^*$. Therefore, $\underline{A}^* < \overline{A}\left(\frac{B}{R_0}Q_k^*\right) \Leftrightarrow \frac{p_H - \frac{1}{2}\frac{R_0}{R}}{\gamma R\omega_H} < V_k^* + \frac{B}{R_0}k^{LB} \Leftrightarrow \frac{1}{\gamma R}\frac{1}{1-p_H} - V_k^* - \frac{B}{R_0}k^{LB} < \frac{R_0}{2\gamma R^2\omega_H} \Leftrightarrow \left(1 - \frac{B}{R_0}\right)k^{LB} < \frac{R_0}{2\gamma R^2\omega_H}$. Because $1 - \frac{B}{R_0} < 0$, it suffices to show $k^{LB} \geq 0$.

For $k = 1$, $k^{LB} = \frac{1}{\gamma R}\frac{1}{1-p_H} - V_1^* = \frac{1}{\gamma R}\frac{1}{1-p_H} - \frac{p^* - \frac{R_0}{R}}{\gamma R \Omega}$. This is positive if $\frac{\Omega}{1-p_H} > p^* - \frac{R_0}{R}$. Using $\Omega = \frac{\omega_H\omega_L}{\lambda\omega_L + (1-\lambda)\omega_H} = \frac{(1-p_H)p_H\omega_L}{\lambda\omega_L+(1-\lambda)\omega_H}$, $\frac{\Omega}{1-p_H} > p^* - \frac{R_0}{R} \Leftrightarrow \frac{p_H\omega_L}{\lambda\omega_L+(1-\lambda)\omega_H} > p^* - \frac{R_0}{R} \Leftrightarrow \frac{p_H\omega_L}{p^* - \frac{R_0}{R}} > \lambda(\omega_L - \omega_H) + \omega_H \Leftrightarrow \frac{\frac{p_H}{p^*-\frac{R_0}{R}}\omega_L - \omega_H}{\omega_L - \omega_H} > \lambda$. The left hand side is greater than one for $R \geq \frac{R_0}{p_L} > \frac{R_0}{p^*}$.

For $k = 2$, $k^{LB} = \frac{1}{\gamma R}\frac{1}{1-p_H} - V_2^* = \frac{1}{\gamma R}\frac{1}{1-p_H} - \lambda\frac{p_H - \frac{R_0}{R}}{\gamma R\omega_H}$. This is positive because $\frac{p_H}{p_H - \frac{R_0}{R}} > \lambda$ for $R \in \left(\frac{R_0}{p_H}, \frac{R_0}{p_L}\right)$.

■

Lemma 8 implies that the unconstrained Q_k^*-equilibrium, $k \in \{1,2\}$, with $Q_k^* = 1$ is in fact Type I-robust for $A > \overline{A}\left(\frac{B}{R_0}\right) = \frac{1}{\gamma R}\frac{1}{1-p_H} + \left(\frac{B}{R_0} - 1\right)k^{LB}$, while it is Type II robust for $A \in \left[\underline{A}^*, \overline{A}\left(\frac{B}{R_0}\right)\right]$. Also, it is not robust for $A < \underline{A}^*$ because borrowing $X^{LB}(Q_k^*)$ and submitting $k^{LB}(Q_k^*)$ would give a higher payoff $\widehat{U}_H > U_H^*(Q_k^*)$ for H-banks.

6 Remaining tasks

So far, we have characterized Q_k^*-equilibrium for $k \in \{1, 2\}$. We need to investigate whether Q_k^*-equilibrium for $k \in \{3, 4\}$ can be Type I robust, and if so, under what conditions. Finally, we need to characterize Q_k^{LB}-equilibrium for $k \in \{1, 2, 3, 4\}$ as shown below:

		Buy orders	
		λk^{LB}	$\lambda k^{LB}(Q)$
Sell	$(1-\lambda)\left(\frac{p_L R - Q R_0}{\gamma \omega_L R^2} - V\right)$	Q_1^{LB}-eqb	Q_3^{LB}-eqb
orders	$-(1-\lambda) V$	Q_2^{LB}-eqb	Q_4^{LB}-eqb

Note that "Q_2^{LB}-eqb" can not generally be a market-clearing equilibrium as no order is price-contingent. For the other cases, a market-clearing condition is well defined.