Efficiency, Equity, and Social Rationality under Uncertainty*

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Abstract

In a simple model where agents' monetary payoffs are uncertain, this paper examines the aggregation of subjective expected utility functions which are interpersonally noncomparable. A maximin social welfare criterion is derived from axioms of efficiency, *ex post* equity, and social rationality, combined with the independence of beliefs and risk preferences in riskless situations (Chambers and Echenique, 2012). The criterion compares allocations by the values of the prospects composed of the statewise minimum payoffs evaluated by the certainty equivalents. Because of this construction, the criterion is egalitarian and uncertainty averse.

1 Introduction

Which social welfare criterion should be adopted to evaluate public policies under uncertainty? In the present paper, we address this question by exploring the implications of equity, efficiency, and social rationality. As these principles are central to the welfare economics of risk and uncertainty, the exploration is important for constructing a reasonable social criterion.[1]

The path-breaking work is Harsanyi's (1955) aggregation theorem. In the context of risk, this states that if individuals and the social observer are expected utility maximizers, the observer's

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[1]For a comprehensive survey, see Mongin and Pivato (2016).

utility function (social welfare criterion) satisfying the standard *ex ante* Pareto principle[2] is represented by the weighted sum of individual expected utilities. The condition of expected utility maximization by the social observer and the *ex ante* Pareto principle are considered as requirements of social rationality and efficiency, respectively.

Harsanyi's result revealed serious tensions between equity (*ex ante* or *ex post*), efficiency, and social rationality. Among others, Diamond (1967) insists that the social expected utility condition is not desirable because it conflicts with *ex ante* equity in the sense of inequality aversion to the distribution of individual expected utilities. Moreover, Grant (1995) shows that *ex ante* egalitarian criteria cannot be compatible with the "minimal" social rationality, *Statewise Dominance*, which requires that if one allocation is *ex post* socially preferred to the other in all states, then the former should be *ex ante* socially preferred to the latter. Fleurbaey and Voorhoeve (2013) argue that social welfare criteria satisfying the *ex ante* Pareto principle cannot help the *ex post* worst-off individual without violating *Statewise Dominance*.

This paper studies requirements of *ex post* equity, efficiency, and social rationality which are compelling and mutually compatible. In particular, given the result by Fleurbaey and Voorhoeve (2013) above, we introduce efficiency conditions that are weaker than the *ex ante* Pareto principle and compatible with a standard *ex post* equity and *Statewise Dominance*. We consider *ex post* equity rather than *ex ante* equity, given that the latter is incompatible with *Statewise Dominance*, which is the minimal requirement for rationality (Grant, 1995).

There are yet other reasons to require weaker efficiency conditions. For one, judgment under uncertainty is difficult for individuals because of heuristics and biases, and it is not compelling to fully respect *ex ante* preferences (Hammond, 1981). Another reason is that when agents' beliefs differ, the *ex ante* Pareto principle and the social expected utility condition are incompatible (Mongin, 1995). Moreover, if individuals have different beliefs, unanimous agreement on uncertain allocations may be spurious because of the disagreement of expectations for future outcomes. This is the problem of spurious unanimity pointed out by Mongin (2016). An example given by Gilboa et al. (2004) illustrates this problem: Two men have decided to fight a duel, and each believes he will win. The *ex ante* Pareto principle would support their agreement, but this seems unreasonable

[2]The *ex ante* Pareto principle asserts that if all agents prefer one allocation of prospects to the other, then the observer also prefers the former.

because one of them will be proven wrong and defeated. We introduce weaker efficiency axioms which avoid these problems. [3]

This paper considers five axioms, which we refer to as *the basic axioms*, in accordance with the above three principles as well as *Invariance of Risk Attitudes and Beliefs for Constant Acts* (henceforth referred to as *IRBC*) introduced by Chambers and Echenique (2012). It is argued that the basic axioms have strong implications on inequality and uncertainty aversion. Although we only require a fairly weak equity condition, we show that the basic axioms imply a property of *strong priority to the worst-off*, which requires that income should be redistributed to the worst-off agent in each state. In the main theorem, we characterize a maximin social welfare criterion by the basic axioms. The criterion assesses each allocation using the prospect composed of the statewise minimum monetary payoffs in the allocation. The value of the prospect is evaluated by the lowest certainty equivalent. This criterion is inequality averse because it focuses on the worst-off agent in each state. It is also uncertainty averse in the sense that it evaluates allocations with the lowest certainty equivalents.

We also argue that the criterion satisfies a separability requirement that an unconcerned individuals who has the same riskless prospect under two allocations should not affect the social judgment over the two allocations. This requirement is desirable in two respects. First, an individual who has the same sure prospect in the allocations could be interpreted one being already dead at the moment of the social evaluation. Then, it is not compelling to take well-being of the dead agent into account, and thus the separability requirement can be justified in terms of "independence of the utilities of the dead" (Blackorby et al., 2005; Fleurbaey, 2010; Fleurbaey and Zuber, 2013; and Fleurbaey et al., 2015). Second, if separability is fully applied to any unconcerned individuals under uncertainty, social evaluations cannot be sensitive to correlations of outcomes among agents (Fleurbaey and Zuber, 2013). Moreover, the combination of full separability and quite weak conditions of equity and efficiency leads to the *ex ante* Pareto principle (Miyagishima, 2016, Lemma 1), and thus the worst-off individual cannot be helped under *Statewise Dominance* by the argument of Fleurbaey and Voorhoeve (2013).

[3] Blume et al. (2014), Brunnermeier et al. (2014) and Gayer et al. (2014) provide other examples. Other recent contributions to this issue include, among others, Gilboa et al. (2014), Danan et al. (2016), Mongin and Pivato (2015), Hayashi and Lombardi (2016).

We adopt a simple economic model where agents' future monetary payoffs are uncertain. It is assumed that agents are subjective expected utility maximizers. The assumption of expected utility would be stringent, because different individuals may follow different principles of decision-making under uncertainty and some of them may be probabilistically unsophisticated. However, our result holds on a broader class of domains and does not depend on the assumption of expected utility.

In this paper, we assume that utilities are ordinally measurable and interpersonally noncomparable. We consider the aggregation of various different expected utilities following the fair social ordering approach (Fleurbaey and Maniquet, 2011; Fleurbaey and Zuber, 2017). Specifically, in this paper, the lowest certainty equivalent is derived as the measure to evaluate allocations.

The most related paper is Fleurbaey and Zuber (2017), who characterized the same social criterion under the assumption that all agents have the same belief as the planner.[4] Our result also holds in their environment. The main differences are as follows. While they used the rationality requirement that the observer should also be a expected utility maximizer, we require *Statewise Dominance*, which is much weaker than the social expected utility condition. Neither continuity nor completeness of social welfare criterion is needed for our main result. Instead, we use a weak efficiency condition, *Pareto for Ex-post Equality*, which requires that an uncertain allocation should be socially preferred to a riskless allocation if the former is an *ex ante* Pareto improvement to the latter and the *ex post* inequality is not larger in the former than in the latter. In other words, *ex ante* Pareto improving risk-takings are socially preferred as long as there is no risk of increasing *ex post* inequalities among equals.

The organization of this paper is as follows. Section 2 presents the model. Section 3 introduces the five basic axioms. Section 4 analyzes the implications of these axioms. Section 5 gives the main theorem characterizing the social criterion using these axioms. In Section 6, we provide some concluding remarks.

[4]They also obtained another characterization of the same criterion in the model where both agents and the planner have maxmin expected utility functions.

2 The Model

Let $N = \{1, ..., n\}$ be the set of agents such that $n \geq 2$. $S = \{s_1, ..., s_m\}$ is the finite set of states with $m \geq 2$. We denote $f_{is} \in \mathbb{R}_+$ the amount of money agent i receives under state $s \in S$. An act of agent i is denoted by $f_i = \left(f_{is}\right)_{s \in S} \in \mathbb{R}_+^S$, which is a vector of state-contingent monetary payoffs. Let $A = \mathbb{R}_+^S$ be the set of acts. $x = \left(x_s\right)_{s \in S} \in A$ is called a *constant act* if $x_s = x_{s'}$ for all $s, s' \in S$. Let $\bar{A}$ be the set of constant acts. We abuse notation in a standard way by denoting the value of money by $\mathcal{A}$ for each $x \in \bar{A}$. An allocation is denoted by $f_N = (f_i)_{i \in N}$. $\mathcal{A} = A^N$ is the set of allocations. Let $\bar{\mathcal{A}} = \bar{A}^N$ be the set of *constant allocations*, which are allocations composed of constant acts. Let us also denote $\mathcal{A}^e = \{f_N \in \mathcal{A} | f_i = f_j \text{ for all } i, j \in N\}$, which is the set of allocations where all agents have the equal acts.

For each $f \in A$ and each $s \in S$, let $f(s) \in \bar{A}$ be such that $f_{s'}(s) = f_{s''}(s) = f_s$ for all $s', s'' \in S$. $f_N(s) \in \bar{\mathcal{A}}$ is similarly defined.

We assume that all agents are expected utility maximizers. Given a continuous and increasing function $u_i : \mathbb{R}_+ \to \mathbb{R}$, a probability vector $p_i = (p_{is})_{s \in S}$ over S, and an act $f_i \in A$, the agent i's subjective expected utility from f_i is defined by

$$E_{p_i}(u_i \circ f_i) = \sum_{s \in S} p_{is} u_i(f_{is}).$$

$E_{p_i} u_i$ denotes agent i's subjective expected utility function. Let $\mathcal{U}$ denote the set of subjective expected utility functions.

A social quasi-ordering function (SQF) $\boldsymbol{R}$ is a mapping that determines a reflexive and transitive binary relation over the set of allocations for every profile of subjective expected utility functions. The domain is denoted by $\mathcal{D} = \mathcal{U}^N$. A typical profile of subjective expected utility functions is $U = (E_{p_i} u_i)_{i \in N}$. Given $U \in \mathcal{D}$, $\boldsymbol{R}(U)$ is a *social quasi-ordering* over $\mathcal{A}$. Also, let $\boldsymbol{P}(U)$ and $\boldsymbol{I}(U)$ be the strict and indifference parts of $\boldsymbol{R}(U)$, respectively.

3 Basic Axioms

We introduce five basic axioms. The first is *Statewise Dominance*, which is often referred to as the *minimal criterion for rational decision*.

Statewise Dominance. For all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

if $f_N(s)\boldsymbol{R}(U)f'_N(s)$ for all $s \in S$, then $f_N\boldsymbol{R}(U)f'_N$, and

if $f_N(s)\boldsymbol{P}(U)f'_N(s)$ for all $s \in S$, then $f_N\boldsymbol{P}(U)f'_N$.

This axiom states that if every outcome of an allocation is socially better than that of another allocation, the former is socially preferred to the latter. If the axiom is violated, society may choose an allocation resulting in a worse consequence.

Next, we introduce two efficiency axioms. The first Pareto axiom takes *ex post* equality into account, and is therefore suitable for our purpose to find a social criterion satisfying *ex post* equity, efficiency, and social rationality. For convenience, we say that f_N is more *ex post* equal than f'_N if $|f_i(s) - f_j(s)| \leq |f'_i(s) - f'_j(s)|$ for all $i, j \in N$ $(i \neq j)$ and $s \in S$, and the strict inequality holds for some $i, j \in N$ $(i \neq j)$ and $s \in S$.

Pareto for Ex-post Equality. For all $U \in \mathcal{D}$ such that $E_{p_i}u_i = E_{p_j}u_j$ for all $i, j \in N$, and all $f_N \in \mathcal{A}$, $x_N \in \bar{\mathcal{A}}$ such that f_N is more *ex post* equal than x_N, if $E_{p_i}(u_i \circ f_i) > u_i(x_i)$ for all $i \in N$, then $f_N\boldsymbol{P}(U)x_N$.

This axiom states that if all agents are willing to take risks (when f_N is uncertain) and the outcomes are more equal than those before the risk-taking, then such risk-taking preferences should be socially supported. When f_N is also constant, the axiom is further compelling because all agents' monetary payoffs increase without any risk. This axiom is reasonable in terms of compatibility with *ex post* equality. Moreover, by the condition that all agents have the same preference, there is a consensus in the sense of Sprumont (2012) that everyone has a better prospect. This Pareto condition also avoids the problem of spurious unanimity caused by different beliefs among agents, because the agents are supposed to have the same belief.

The next efficiency condition is introduced by Fleurbaey and Zuber (2017).

Pareto for Equal or No Risk. For all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}^e \cup \bar{\mathcal{A}}$, if $E_{p_i}(u_i \circ f_i) > E_{p_i}(u_i \circ f'_i)$ for all $i \in N$, then $f_N\boldsymbol{P}(U)f'_N$.

When comparing uncertain allocations where all agents have equal acts and are therefore under the egalitarian condition, it is compelling to judge that the unanimously preferred allocation should be more socially desirable (Fleurbaey, 2010). If allocations are constant, unanimous improvements are also socially desirable. *Pareto for Equal or No Risk* combines these ideas, but is still much weaker than the *ex ante* Pareto principle. This axiom also avoids spurious unanimity because all agents have the same acts under the uncertain allocations.

Next, we introduce an equity condition.

Ex-post Transfer among Equals. For all $U \in \mathcal{D}$ and all $x_N, x'_N \in \bar{\mathcal{A}}$, if there exist j, k such that $E_{p_j} u_j = E_{p_k} u_k$, and $x_i = x'_i$ for all $i \neq j, k$, then for all $t > 0$,

$$\left[x_j = x'_j - t > x_k = x'_k + t\right] \Rightarrow x_N \boldsymbol{R}(U) x'_N.$$

This axiom requires that if there is an *ex post* inequality between two agents with the same preference, it should be socially accepted to reduce the inequality by transfers. The restriction to individuals with the same preference is meaningful in terms of equal treatment of equals.

The next invariance axiom is essentially based on that introduced by Chambers and Echenique (2012).

Invariance to Risk Attitudes and Beliefs for Constant Acts (IRBC). For all $U = (E_{p_i} u_i)_{i \in N}$, $U' = (E_{p'_i} u'_i)_{i \in N} \in \mathcal{D}$ and all $x_N, x'_N \in \bar{\mathcal{A}}$, $x_N \boldsymbol{R}(U) x'_N$ if and only if $x_N \boldsymbol{R}(U') x'_N$.

This axiom claims that social judgements over constant allocations should be invariant of risk preferences and beliefs. The idea is that as long as riskless outcomes are compared, agents' risk preferences are irrelevant for the comparisons.

4 Implications of the Basic Axioms

In this section, we derive the implications of our basic axioms. These implications are not only interesting in their own right, but also useful to prove our main theorem.

The first lemma states that *Ex-post Transfer among Equals*, *Statewise Dominance*, and *IRBC* together imply the following strong equity axiom.

Transfer. For all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$, if there exist j, k such that $f_i = f'_i$ for all $i \neq j, k$, then for all $\Delta \in \mathbb{R}^S_{++}$,

$$\left[f_j = f'_j - \Delta \gg f_k = f'_k + \Delta\right] \Rightarrow f_N \boldsymbol{R}(U) f'_N.$$

This axiom states that for two agents, if one has more income in every state than the other, a transfer in each state to reduce the inequality should be acceptable.

Lemma 1. *Ex-post Transfer among Equals*, *IRBC*, and *Statewise Dominance* together imply *Transfer*.

Proof. Let $f_N, f'_N \in \mathcal{A}$ be such that $f_i = f'_i$ for all $i \neq j, k$, and

$$f_j = f'_j - \Delta \gg f_k = f'_k + \Delta,\ \Delta \in \mathbb{R}^S_{++}.$$

Consider $f_N(s), f'_N(s) \in \bar{\mathcal{A}}$ for each $s \in S$. Let U' be such that $E_{p_i} u_i = E_{p_j} u_j$ for all $i, j \in N$. By assumption and *Ex-post Transfer among Equals*, we have $f_N(s)\boldsymbol{R}(U')f'_N(s)$ for all $s \in S$. It follows from *IRBC* that $f_N(s)\boldsymbol{R}(U)f'_N(s)$ for all $s \in S$. Then, $f_N\boldsymbol{R}(U)f'_N$ follows from *Statewise Dominance*. □

Lemma 1 has an important normative implication. From the fundamental incompatibility of equity and efficiency shown by Fleurbaey and Trannoy (2003), we can see that there is no SQF satisfying both *Transfer* and the *ex ante* Pareto principle. Thus, we have to give up the *ex ante* Pareto if *IRBC* and *Statewise Dominance* are required in addition to the weak equity condition, *Ex-post Transfer among Equals*. Intuitively, differences in preference become irrelevant for the equity axiom by *IRBC*, and transfers among uncertain prospects become favorable for the society by *Statewise Dominance*.

The next lemma shows that social criteria satisfying *Pareto for Equal or No Risk* and *Statewise Dominance* are monotonically increasing.

Lemma 2. If $\boldsymbol{R}$ satisfies *Pareto for Equal or No Risk* and *Statewise Dominance*, then for all $E_p u_N \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

$$f_N \gg f'_N \Longrightarrow f_N\boldsymbol{P}(U)f'_N.$$

Proof. Let $f_N, f'_N \in \mathcal{A}$ be such that $f_N \gg f'_N$. Since $f_i(s) \gg f'_i(s)$ for all $i \in N$ and all $s \in S$, *Pareto for Equal or No Risk* implies $f_N(s)\boldsymbol{P}(U)f_N(s)$ for all $s \in S$. The desired conclusion follows from *Statewise Dominance*. □

The next lemma establishes an infinite *ex post* inequality aversion, which is captured by the following axiom.

Strong Priority to the Worst-off. For all $U \in \mathcal{D}$ and all $x_N, x'_N \in \bar{\mathcal{A}}$, if there exist $j, k \in N$ such that $x_k = \min_{i \in N} x_i$, $x'_k = \min_{i \in N} x'_i$, and $x_i = x'_i$ for all $i \neq j, k$, then

$$\left[x'_j > x_j > x_k > x'_k\right] \Rightarrow x_N \boldsymbol{P}(U) x'_N.$$

Lemma 3. The basic axioms imply *Strong Priority to the Worst-off.*

Proof. Since x_N and x'_N are constant allocations, we can invoke *IRBC* to arbitrarily modify the subjective expected utility functions. Then, suppose that all agents have the common expected utility function $E_{p_0} u_0$ defined below. Let $x''_N \in \bar{\mathcal{A}}$ be such that $x'_k < x''_k < x_k$, and given $\bar{x} > \max_{i \in N} x'_i$, $x''_h = \bar{x}$ for all $h \neq k$. Let $a, b, c, \alpha, \beta, \gamma, \delta, \epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}_{++}$ be parameters such that

$$\alpha x_k + \beta(a - x_k) = m\alpha x''_k + \epsilon_1, \tag{1}$$

$$\alpha x_k + \beta(a - x_k) + \gamma(b - a) = m\alpha x_k - \epsilon_2, \tag{2}$$

$$\alpha x_k + \beta(a - x_k) + \gamma(b - a) + \delta(c - b) = m[\alpha x_k + \beta(x''_j - x_k)] + \epsilon_3, \tag{3}$$

where $a > x_k$, $c - a = x''_j - x''_k$, $b = \frac{(n-1)c + a}{n}$, and $\epsilon_1, \epsilon_2, \epsilon_3$ are arbitrarily close to 0. By simple calculations, we have

$$\beta = \frac{m\alpha x''_k - \alpha x_k + \epsilon_1}{a - x_k},$$
$$\gamma = \frac{m\alpha(x_k - x''_k) - \epsilon_1 - \epsilon_2}{b - a},$$
$$\delta = \frac{\beta(x''_j - x_k) - \epsilon_2 + \epsilon_3}{c - b}.$$

Note that

$$b - a = \frac{1}{n}(x''_j - x''_k), \ c - b = \frac{n-1}{n}(x''_j - x''_k).$$

For $\alpha, \beta, \gamma, \delta$ to be positive, we can set, for instance,

$$a = 2x_k, \ \alpha = \frac{\epsilon_1}{2|m x''_k - x_k|}.$$

Now, we can define $u_0 : \mathbb{R}_+ \to \mathbb{R}_+$ as follows.

$$u_0(x) = \begin{cases} \alpha x \text{ for } x \in [0, x_k], \\ \alpha x_k + \beta(x - x_k) \text{ for } x \in (x_k, a], \\ \alpha x_k + \beta(a - x_k) + \gamma(x - a) \text{ for } x \in (a, b], \\ \alpha x_k + \beta(a - x_k) + \gamma(b - a) + \delta(x - b) \text{ for } x > b, \end{cases}$$

Let us also define $p_{0s} = 1/m$ for all $s \in S$.[5] Then, $E_{p_0} u_0$ is defined by p_0 and u_0. It is straightforward to check that this is consistent with conditions (1) to (3) above. Note that $E_{p_0} u_0$ satisfies

$$\frac{u_0(a)}{m} > u_0(x''_k), \ \frac{u_0(b)}{m} < u_0(x_k), \ \frac{u_0(c)}{m} > u_0(x''_j).$$

Let $U \in \mathcal{D}$ denote the preference profile where all agents have $E_{p_0} u_0$.

We have $x''_N \boldsymbol{P}(U) x'_n$ by *Pareto for Equal or No Risk*. In the following, we show $x_N \boldsymbol{P}(U) x''_N$, which implies $x_N \boldsymbol{P}(U) x'_N$ by transitivity. Then, we can complete the proof by adjusting the preferences using *IRBC*.

From the construction of $E_{p_0} u_0$, there exist $f^a, g^b, f^c \in A$ which are arbitrarily close to $(a, 0, \cdots, 0), (b, 0, \cdots, 0), (c, 0, \cdots, 0) \in A$ respectively, and satisfy

$$g^b = \frac{n-1}{n} f^c + \frac{1}{n} f^a,$$
$$E_{p_0}(u_0 \circ f^a) > u_0(x''_k), \ E_{p_0}(u_0 \circ g^b) < u_0(x_k), \ E_{p_0}(u_0 \circ f^c) > u_0(x''_j).$$

Define $f_N, g_N, g'_N \in \mathcal{A}$ such that

$$f_k = f^a, \ f_i = f^c \text{ for all } i \neq k,$$
$$g_k = g^b - \boldsymbol{\epsilon}, \ g_i = g^b + \boldsymbol{\epsilon} \text{ for all } i \neq k,$$
$$g'_i = g^b + 2\boldsymbol{\epsilon} \text{ for all } i \in N,$$

where $\boldsymbol{\epsilon} \in \mathbb{R}^S_{++}$ is small enough that

$$g_k \gg f_k, \ E_{p_0}(u_0 \circ g'_i) < u_0(x_k).$$

Then, *Pareto for Equal or No Risk* implies $f_N \boldsymbol{P}(U) x''_N$. By repeated applications of *Transfer* (Lemma 1), we have $g_N \boldsymbol{R}(U) f_N$. Lemma 2 implies $g'_N \boldsymbol{P}(U) g_N$. Moreover, $x_N \boldsymbol{P}(U) g'_N$ follows

[5] Remember that m is the cardinality of S.

from *Pareto for Equal or No Risk*. By transitivity, we obtain $x_N \boldsymbol{P}(U) x''_N$, as sought. $\square$

The next lemma is useful to prove our main theorem. For each $f_N \in \mathcal{A}$, we denote $m(f_N) = (\min_{i \in N} f_{is})_{s \in S} \in A$, which is the prospect composed of the statewise minimum payoffs in f_N.

Lemma 4. Suppose that $\boldsymbol{R}$ satisfies the basic axioms. Then, for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

$m(f_N) \gg m(f'_N)$ implies $f_N \boldsymbol{P}(U) f'_N$.

Proof. Suppose $m(f_N) \gg m(f'_N)$. Consider $f_N(s), f'_N(s) \in \bar{X}$ for each $s \in S$. Note that by assumption,

$$\min_{i \in N} f_i(s) > \min_{i \in N} f'_i(s) \text{ for every } s \in S.$$

For our purpose, it is sufficient to show $f_N(s) \boldsymbol{P}(U) f'_N(s)$ for every $s \in S$, which implies $f_N \boldsymbol{P}(U) f'_N$ by *Stetewise Dominance*, as sought.

The rest of the proof is divided into two cases.

Case 1. Suppose $f'_j(s) = \min_{i \in N} f'_i(s)$ for all $j \in N$. Then,

$$f_j(s) \geq \min_{i \in N} f_i(s) > f'_j(s) \text{ for all } j \in N,$$

and hence $f_N(s) \boldsymbol{P}(U) f'_N(s)$ by *Pareto for Equal or No Risk*.

Case 2. Suppose $f'_j(s) > \min_{i \in N} f'_i(s)$ for some $j \in N$. Let $x_N \in \bar{X}$ be such that

$$x_j = \frac{\min_{i \in N} f_i(s) + \min_{i \in N} f'_i(s)}{2} \text{ for all } j \in N,$$

Then, it is straightforward to show that repeated applications of Lemma 3, *Pareto for Equal or No Risk*, and transitivity together imply $x_N \boldsymbol{P}(U) f'_N(s)$. $f_N(s) \boldsymbol{P}(U) x_N$ follows from *Pareto for Equal or No Risk*. We obtain $f_N(s) \boldsymbol{P}(U) f'_N(s)$ by transitivity. $\square$

While *Ex-post Transfer among Equals* is quite weak, if it is combined with other basic axioms, the social criterion should be sensitive to the statewise worst-off individuals. The statewise worst-offs are crucial for our main theorem in the next section.

5 The Social Criterion and Its Characterization

In this section, we derive the social welfare criterion from the basic axioms. For convenience, we introduce a notation. Given $f_i \in A$ and $E_{p_i}u_i \in \mathcal{U}$, let

$$C(f_i, E_{p_i}u_i) = \inf\{c \in \mathbb{R}_+ | u_i(c) \geq E_{p_i}(u_i \circ f_i)\},$$

which is the certainty equivalent of f_i with respect to $E_{p_i}u_i$.

Then, we obtain the following result.

Theorem. Suppose that an SQF $\boldsymbol{R}$ satisfies the basic five axioms. Then, for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

$$\min_{i\in N} C\big(m(f_N), E_{p_i}u_i\big) > \min_{i\in N} C\big(m(f'_N), E_{p_i}u_i\big) \Longrightarrow f_N \boldsymbol{P}(U) f'_N.$$

The social criterion evaluates each allocation f_N by $m(f_N)$ based on the minimum value of certainty equivalents among individuals.

Proof of Theorem. Let f_N and f'_N be allocations satisfying the condition of the theorem. Without loss of generality, assume that $C(m(f'_N), E_{p_1}u_1) = \min_{i\in N} C\big(m(f'_N), E_{p_i}u_i\big)$. Consider $\boldsymbol{\epsilon} = (\epsilon, \cdots, \epsilon) \in \mathbb{R}^S_{++}$ and $g_N, g'_N \in X$ such that

$$g'_i = m(g'_N) = m(f'_N) + \boldsymbol{\epsilon} \text{ for all } i \in N,$$

$$g_i = m(g_N) = m(f_N) - \boldsymbol{\epsilon} \text{ for all } i \in N,$$

$$\min_{j\in N} C\big(m(g_N), E_{p_j}u_j\big) > \min_{j\in N} C\big(m(g'_N), E_{p_j}u_j\big) + 3\epsilon.$$

By Lemma 4, we have $f_N \boldsymbol{P}(U) g_N$ and $g'_N \boldsymbol{P}(U) f'_N$. In the following, we show $g_N \boldsymbol{P}(U) g'_N$. Then, by transitivity, we have the desired result.

Consider $x_N, y_N \in \bar{X}$ such that

$$x_1 = \min_{j\in N} C\big(m(g'_N), E_{p_j}u_j\big) + \epsilon,\ u_i(x_i) > E_{p_i}(u_i \circ g'_i),\ x_i > x_1 + 3\epsilon \text{ for all } i \neq 1,$$

$$y_1 = x_1 + 2\epsilon,\ y_i = x_1 + 3\epsilon \text{ for all } i \neq 1.$$

By *Pareto for Equal or No Risk*, we obtain $x_N \boldsymbol{P}(U) g'_N$. It follows from repeated applications of *Strong Ex-post Inequality Aversion* (Lemma 3) that $y_N \boldsymbol{P}(U) x_N$. Note that $m(g_N) = g_i$ and

$E_{p_i}(u_i \circ g_i) > y_i$ for all $i \in N$. Hence, *Pareto for Equal or No Risk* implies $g_N \boldsymbol{P}(U) y_N$. By transitivity, we obtain $g_N \boldsymbol{P}(U) g'_N$ as sought. $\square$

Note that the characterization is partial. If the standard continuity is additionally required, we can obtain the full characterization of the following social criterion.

Definition. $\boldsymbol{R}_M$ is a social ordering function such that for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

$$f_N \boldsymbol{R}_M(U) f'_N \Longleftrightarrow \min_{i \in N} C\big(m(f_N), E_{p_i} u_i\big) \geq \min_{i \in N} C\big(m(f'_N), E_{p_i} u_i\big).$$

This criterion is inequality averse because it focuses on the worst-off agent in each state. This criterion is also uncertainty averse in the sense that it evaluates allocations using the lowest certainty equivalents. Note that completeness of the social criterion is obtained from the result. Moreover, this criterion satisfies expected utility conditions when the social observer and agents share the same belief. This stronger rationality condition is obtained from the characterization.

The basic axioms are satisfied also by a leximin criterion. To introduce the leximin criterion, we introduce several notations. Given $f_N \in \mathcal{A}$ and $s \in S$, $f_{(i)s}$ is the ith lowest income in $f_N(s)$.[6] Let us denote $m_{(i)}(f_N) = (f_{(i)s})_{s \in S}$, which is the prospect composed of the ith lowest monetary payoffs in f_N.

Definition. $\boldsymbol{R}_{LM}$ is a social ordering function such that for all $U \in \mathcal{D}$ and all $f_N, f'_N \in \mathcal{A}$,

$$f_N \boldsymbol{R}_{LM}(U) f'_N \Longleftrightarrow \Big(\min_{i \in N} C\big(m_{(j)}(f_N), E_{p_i} u_i\big)\Big)_{j \in N} \geq_{lex} \Big(\min_{i \in N} C\big(m_{(j)}(f'_N), E_{p_i} u_i\big)\Big)_{j \in N},$$

where $\geq_{lex}$ the standard lexicographic ordering.

It is worth discussing that the basic axioms and separability requirements are consistent. Separability for sure prospects is important for social evaluation under uncertainty, because the requirement is interpreted as independence of the utility of the dead (Bossert et al., 2005). It would not be reasonable if social evaluations are influenced by utilities of agents who do not exist (Fleurbaey, 2010; Fleurbaey and Zuber, 2013). This idea is captured by the following axiom.

Separability for Sure Prospects. For all $U \in \mathcal{D}$, all $x_N, x'_N \in \bar{A}$, if $x_i = x'_i$ for some $i \in N$, then for all $y_i \in \bar{A}$,

$$x_N \boldsymbol{R}(R_N) x'_N \Longleftrightarrow (x_{N \setminus \{i\}}, y_i) \boldsymbol{R}(U) (x'_{N \setminus \{i\}}, y_i).$$

[6] Ties can be broken arbitrarily.

This axiom requires that an agent should not affect the evaluation of *constant allocations* if the agent has the same act in the allocations. As mentioned in the introduction, it is important to restrict the application of separability to riskless situations in order to be consistent with equity. In dynamic situations, this requirement can be interpreted as independence of the utility of the dead. $\boldsymbol{R}_{LM}$ satisfies an important condition of separability under no risk, and thus our basic axioms are consistent with it.

$\boldsymbol{R}_M$ also satisfies the following weaker separability.

Well-off Separability for Sure Prospects. For all $U \in \mathcal{D}$, all $x_N, x'_N \in \bar{\mathcal{A}}$, if

$$x_i = x'_i > \max\{\min_{i\in N} x_i, \min_{i\in N} x'_i\} \text{ for some } i \in N,$$

then for all $y_i > \max\{\min_{i\in N} x_i, \min_{i\in N} x'_i\}$,

$$x_N \boldsymbol{R}(U) x'_N \Longleftrightarrow (x_{N\setminus\{i\}}, y_i) \boldsymbol{P}(U) (x'_{N\setminus\{i\}}, y_i).$$

This axiom states that when evaluating constant allocations, an unconcerned individual cannot affect the evaluation as long as the agent has larger monetary payoffs than the worst-offs. From an egalitarian viewpoint, the information on the worst-offs is important and the social evaluation may well change if the situations of the worst-offs vary. This axiom captures the idea and restricts the separability principle to the case where the unconcerned agent is not the worst-off.

6 Concluding Remarks

In this paper, we analyzed the implications of equity, efficiency and social rationality under uncertainty. We obtained the social criterion which is sensitive to the statewise worst-off individuals. In the literature of welfare economics under risk and uncertainty, it is an important issue to construct a social welfare criterion satisfying separability and the three principles above (Fleurbaey, 2010; Fleurbaey and Zuber, 2013; Fleurbaey et al., 2015). Our result provided an answer to the problem by deriving the social welfare criterion as a reasonable compromise between equity, efficiency, and social rationality.

To make our analysis simple, we considered the model where each agent's *ex post* well-being is measured in monetary terms and thus single dimensional. The analysis can be extended to the case where agents have preferences over multidimensional outcomes, following the approach developed by Fleurbaey and Zuber (2017). In that case, a criterion for interpersonal comparison is adopted to evaluate well-being *ex post* using the fair social ordering approach (Fleurbaey and Maniquet, 2011).

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