A revenue-enhancing effect of a buyout price in multi-unit uniform-price auctions

Toshihiro Tsuchihashi*

October 2016

Abstract

A demand reduction is prominent and prevalent in multi-unit auctions. Buyers lower a bid on a lesser-valued unit and may cause low seller revenue. We introduce a two-stage model to study multi-unit uniform-price auctions with a buyout price where two buyers with two-unit demand have a buyout option. In the first stage, buyers simultaneously choose a quantity of items they wish to buy at a buyout price. In the second stage, if any, buyer(s) compete in bid. We focus on a situation in which buyers submit a single-unit bid if they have a chance to bid. A single-unit bid yields a zero revenue to a seller. We derive a symmetric undominated Bayesian equilibrium and show that a buyout price improves seller revenue. The revenue-enhancing effect of a buyout price arises from a trade-off faced to buyers between the benefit from obtaining two units at a given buyout price and the cost of losing a chance to win one unit at zero price.

1 Introduction

We frequently observe many items are auctioned for sale at a time in the actual world. These auctions are called multi-unit auctions. The examples of multi-unit auctions include Treasury bill auctions, spectrum auctions, and online auctions such as eBay and Yahoo.

As Vickrey (1961) already pointed in his seminal work, the results obtained in single-unit auctions do not generally apply to multi-unit auctions. Although second-price sealed-bid auctions in the single-unit environment are naturally extended to sealed-bid uniform-price auctions in the multi-unit environment, bidders do not have a dominant strategy in uniform-price auctions. Generally, bidding sincerely their actual valuations on all the units does not even constitute a symmetric equilibrium.

Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) study uniform-price sealed-bid auctions with independent private values in which buyers each demand two units of an item and value the second unit lower than the first unit. Noussair (1995) shows that an equilibrium bid on one unit is irrelevant to the valuation of the other unit. Specifically, in a symmetric undominated equilibrium, buyers sincerely bid on the first unit, i.e., a first-unit bid is equal to a first-unit valuation. A sincere bid on the first unit is intuitive because a first-unit bid has no impact on the payment for the first unit, similarly with a dominant strategy in second-price sealed-bid auctions.

*Faculty of Economics, Daito Bunka University; 1-9-1 Takashimadaira, Itabashi-ku Tokyo, 175-8571 Japan; E-mail:tsuchihasi@ic.daito.ac.jp

On the other hand, a buyer has an incentive to lower a bid on the second unit because his second-unit bid determines the payment for the first unit with a positive probability. A buyer then faces a trade-off between the benefit from lowering an expected payment and the cost of decreasing a winning probability. Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) show that the benefit dominates the cost; that is, in a symmetric undominated equilibrium, a second-unit bid is chosen below a second-unit valuation.

This bidding strategy, referred to as a *demand reduction*, is optimal in more general environments involving many buyers who demand more than two units. Ausubel and Cramton (1996) and Ausubel et al. (2014) consider uniform-price sealed-bid auctions in which infinitely divisible items are auctioned and show that a buyer optimally lowers bids on lesser-valued units if at least one buyer has downward-sloping demand.

A demand reduction yields inefficient allocations and lowers seller revenue. In the extreme, all units are sold for a zero price. Engelbrecht-Wiggans and Kahn (1998) provide a necessary condition for a buyer with two-unit demand to submit zero as a second-unit bid, i.e., a *single-unit bid*. In this single-unit bid equilibrium, seller revenue falls to zero as long as excess demand occurs.

Experimental studies confirm a demand reduction in uniform-price sealed-bid auctions. Kagel and Levin (2001) conduct a laboratory experiment of multi-unit auctions in which a subject with two-unit demand competes in bid with a computer with single-unit demand. The experimenters program the computer to follow a dominant strategy. This environment theoretically leads to a demand reduction because one buyer (i.e., the computer) has downward-sloping demand. As theory predicts, they find that a buyer lowers a second-unit bid in uniform-price sealed-bid auctions as compared with dynamic Vickrey auctions (or Ausubel auctions) in which buyers optimally reveal their actual demand. Similarly, List and Lucking-Reiley (2000) find an evidence of a demand reduction and a single-unit bid in a field experiment where two sports trading cards are sold against two buyers with two-unit demand. They report significantly high likelihood of a demand reduction and a single-unit bid in uniform-price sealed-bid auctions as compared with sealed-bid Vickrey auctions. These results are confirmed in Porter and Vragov (2006).[1] Engelbrecht-Wiggans et al. (2006) obtain the similar result in the experiment where more bidders are engaged in. Moreover, these studies in common report overbidding on the first unit.

A demand reduction matters in the real world multi-unit auctions. Ausubel and Cramton (1996) discuss strategic similarity between uniform-price sealed-bid auctions and the Federal Communications Commission (FCC) spectrum auctions and suggest that the outcome of the FCC spectrum auctions

[1] The important exception is Alsemgeest et al. (1998). They do not observe a demand reduction in uniform-price sealed-bid auctions but in multi-unit English clock auctions.

show a demand reduction.[2]

It seems valuable to consider ways to deter a demand reduction in uniform-price sealed-bid auctions. In this paper, we suggest that introducing a buyout option can prevent a demand reduction from occurring and hence improves seller revenue. We use a simple two-stage model in which two units of an item are sold at auction to two buyers with two-unit demand. Buyers simultaneously decide whether to exercise a buyout option in the first stage (a buyout stage), and then compete in bid in the second stage (a bid stage), a uniform-price sealed-bid auction, unless the items are sold in the first stage. Our analysis focuses on an equilibrium in which buyers submit a single-unit bid in the second stage and hence each obtain one unit at a zero price. Such a buyout option shows a trade-off faced to buyers between the benefit from obtaining two units at a given buyout price and the cost of losing a chance to win one unit at a zero price.

This paper therefore has a contribution to the literature of buyout prices as well. To the best of my knowledge, all the existing studies investigate a buyout price in multi-unit auctions.[3] In the literature, theoretical studies suggest that a buyout price enhances seller revenue if buyers are risk averse (Budish and Takeyama, 2001; Reynolds and Wooders, 2009) or impatient in time (Mathews, 2004; Mathews and Katzman, 2006).[4] This paper provides other explanation to the literature that a buyout price increases seller revenue.

The rest of this paper is organized as follows. Section 2 models uniform-price auctions with a buyout price as a two-stage game. Two buyers exercise a buyout option in a buyout stage and then compete in bid in a bid stage. Section 3 considers a buyer's optimal decision on bidding in a bid stage. We focus on a single-unit bid. Section 4 investigates a buyer's optimal decision on exercising a buyout option and derives a symmetric undominated Bayesian equilibrium. We see a revenue-enhancing effect of a buyout option in the equilibrium. Section 5 provides a flat-demand example. Section 6 discusses limitations of the analysis and then directs the future research.

2 The model

We model a uniform-price auction with an exogenous buyout price. We focus on a simple situation where two identical items are demanded by two risk-neutral buyers with two-unit demand. We assume no reserve prices. A buyer has a quasi-linear payoff function. That is, given payment p per unit, a

[2] Grimm et al. (2003) report a demand reduction in the second-generation (GSM) spectrum auction in Germany which is a simultaneous ascending-bid multi-unit auction. A demand reduction theoretically constitutes a symmetric equilibrium in simultaneous ascending auctions (Engelbrecht-Wiggans and Kahn, 2005).

[3] There exists a paper that study a buyout price in terms of single-unit auctions with multi-unit *demand*. Kirkegaard and Overgaard (2008) consider a situation where buyers with two-unit demand sequentially participate in two second-price sealed-bid single-unit auctions. In their model, the first seller is solely allowed to post a buyout price. They show that introducing a buyout option improves the first seller revenue whereas reduces revenue of the second seller.

[4] The literature suggests other theoretical explanations about a rational usage of a buyout price. Tsuchihashi (2015) provides a comprehensive survey of the buyout price literature.

payoff of a buyer who values kth unit at x_k $(k = 1, 2)$ is given by $x_1 + x_2 - 2p$ if he obtains two units whereas his payoff is $x_1 - p$ if he obtains one unit. Each buyer's valuations (x_1, x_2) are independently and identically drawn from $X = \{(x_1, x_2) \in [0, \bar{x}]^2 | x_1 \geq x_2\}$ according to a joint distribution function $F(x_1, x_2)$. Moreover, we let $f(x_1, x_2) = \frac{\partial^2 F}{\partial x_1 \partial x_2}(x_1, x_2)$ denote the corresponding density function and assume $f(x_1, x_2) > 0$ for any $(x_1, x_2) \in X$ (i.e., $F(x_1, x_2)$ has a full support).

The auction consists of two stages: a *buyout stage* and a *bid stage*. In a buyout stage, given buyout price b, two buyers simultaneously choose quantity of items q they want to buy at the buyout price, $q \in \{0, 1, 2\}$. If both of two buyers choose a positive quantity and the total quantity exceeds two, items are randomly assigned to buyers in proportion as quantities they choose. Suppose, for example, that buyer 1 and buyer 2 choose $q = 2$ and $q = 1$, respectively. Then, buyer 1 obtains two units and one unit with probability $1/3$ and $2/3$, respectively. On the other hand, buyer 2 obtains one unit and nothing with probability $2/3$ and $1/3$, respectively. If the total quantity equals two, buyer(s) purchase items at the buyout price. In these cases, the game ends with the current stage. That is, a bid stage does not appear. If one buyer solely exercises a buyout option for one unit, the buyer obtains one unit by paying the buyout price and then exists the auction. The other buyer remains and the game proceeds to a bid stage. On the other hand, both buyers choose zero quantity (i.e., they do not exercise a buyout option), then the game proceeds to a bid stage.

In a bid stage, if both two buyers remain, they participate a uniform-price sealed-bid auction and simultaneously submit two bids (p_1, p_2) where p_k represents a bid on the kth unit $(k = 1, 2)$. The selling price is determined at the highest rejected bid. On the other hand, if one buyer solely appears a bid stage, he is awarded one unit at zero price with certainty.[5] Note that this situation happens when either one of the two buyers buys one unit in a buyout stage. By using this model, we consider a buyer's optimal decision on exercising a buyout option and bidding in a symmetric undominated Bayesian equilibrium.

3 Optimal bid

As always, we derive an equilibrium by backward induction and thus first consider a bidding behavior in a bid stage. Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) find that a demand reduction generally occurs in uniform-price auctions. Specifically, it is optimal for a buyer to bid sincerely on the first unit, i.e., $p_1 = x_1$, in a symmetric undominated equilibrium. However, a buyer optimally lowers a bid on the second unit, i.e., $p_2 \leq x_2$. Engelbrecht-Wiggans and Kahn (1998) show that, as

[5]This situation equates the bid stage to a second-price sealed-bid auction in which a single buyer is involved. Note that the "second highest" bid is zero. Such a non-competitive situation is sometimes observed in online auctions. A seller sets a starting price at the lowest level allowed in the system (e.g., one yen in Yahoo auctions in Japan), and an item is indeed sold at the starting price (perhaps against the seller's intention).

the extreme case, there exists a *single-unit bid* equilibrium in which buyers submit a zero bid on the second unit, i.e., $p_2 = 0$. In this equilibrium, seller revenue falls to zero because both two units are sold for a zero price.

Since this paper is motivated by low revenue observed in multi-unit auctions, zero revenue obtained in a single-unit bid equilibrium becomes a benchmark. Therefore, in what follows, we restrict our attention to an equilibrium in which a buyer with valuation (x_1, x_2) submits single-unit bid $(p_1, p_2) = (x_1, 0)$ in a bid stage. Let X_q denote a set of valuations that a buyer chooses quantity q, $X = \cup_{q=0}^{2} X_q$.

Lemma 1. Let $F_1(x_1|X_0)$ denote a marginal distribution conditional on choosing $q = 0$ in a bid stage. Precisely,

$$F_1(x_1|X_0) = \int_{x_2 \in X_0} f(x_1, x_2) dx_2.$$

Suppose that there is zero probability of ties that buyer 1's second-unit bid is equal to buyer 2's first-unit bid. Given X_0, single-unit bid $(p_1, p_2) = (x_1, 0)$ constitutes a symmetric undominated Bayesian equilibrium if for all $p \le \bar{x}$,

$$(\bar{x} - p)\frac{f_1(p|X_0)}{1 - F_1(p|X_0)} \le 1. \tag{1}$$

Proof. Since it is clear that single-unit bid is optimal if a buyer solely appears in a bid stage, we restrict our attention to a bid stage where two buyers involve in. Without a loss of generality, we focus on a bidding behavior of buyer 1 with valuation (x_1, x_2). As Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) show, in any symmetric undominated Bayesian equilibrium, both buyers sincerely bid on the first unit, i.e., $p_1 = x_1$. Thus, we let (x_1, p) be buyer 1's bid. Note that $p \le x_2 \le x_1$ should hold. Suppose that buyer 2 with valuations (y_1, y_2) submits single-unit bid $(p_1, p_2) = (y_1, 0)$. Since the two buyers appear in a bid stage if and only if they choose $q = 0$ in a buyout stage, buyer 2's valuations are belonging to X_0.

As in Engelbrecht-Wiggans and Kahn (1998), by ignoring cases of $y_1 = p$, there are three cases: (i) $y_1 < p$, (ii) $p < y_1 < x_1$, and (iii) $y_1 \ge x_1$. If $y_1 < p$, buyer 1 wins two units and pays the highest rejected bid y_1 per unit. Otherwise, buyer 1 obtains one unit and pays his own second-unit bid p. Let $F_1(y_1|X_0)$ denote the marginal distribution of buyer 2's first-unit valuation conditional on choosing $q = 0$. The conditional expected payoff of buyer 1 to submit $(p_1, p_2) = (x_1, p)$ is given by

$$\begin{aligned}
\pi(p, x_1, x_2) &= \int_0^p (x_1 + x_2 - 2y_1) dF_1(y_1|X_0) + \int_p^{\bar{x}} (x_1 - p) dF_1(y_1|X_0) \\
&= (x_1 + x_2)F_1(p|X_0) - 2\int_0^p y_1 dF_1(y_1|X_0) + (x_1 - p)[1 - F_1(p|X_0)] \\
&= (x_2 - p)F_1(p|X_0) + 2\int_0^p F_1(y_1|X_0) dy_1 + (x_1 - p).
\end{aligned}$$

The third equality arises from integration by parts. By differentiating the expected payoff with respect to p, we obtain

$$\frac{\partial \pi}{\partial p}(p, x_1, x_2) = -[1 - F_1(p|X_0)] + (x_2 - p) f_1(p|X_0).$$

Given X_0, buyer 1 optimally chooses $p = 0$ if for any $x_2 \in [0, \bar{x}]$ and $p \leq x_2$,

$$-[1 - F_1(p|X_0)] + (\bar{x} - p) f_1(p|X_0) \leq 0,$$

which is rewritten as Equation (1). □

Lemma 1 is a corollary of Theorem 4.2 appeared in Engelbrecht-Wiggans and Kahn (1998) because they consider a case of n buyers. Note that Theorem 4.2 in Engelbrecht-Wiggans and Kahn (1998) requires that the number of buyers is at least as large as the number of items, and this requirement is satisfied in our setting. We can interpret Equation (1) as a condition on conditional marginal distribution $F_1(x_1|X_0)$, which depends on X_0. For example, if $F_1(x_1|X_0)$ is uniform, then Equation (1) is satisfied for any X_0 because $f_1(x_1|X_0) = 0$ for any $x_1 \in [0, \bar{x}]$. Generally, Equation (1) is more likely to be satisfied for larger X_0. To obtain the underlying intuition, suppose contrarily that X_0 is sufficiently small. In this case, if a bid stage emerges and both buyers appear there, an opponent must have sufficiently low valuations that locate in a "left-lower" region in a (x_1, x_2)-plane. Thus, an incentive to submit a single-unit bid is weakened because a buyer may probably obtain two units at a low price by raising a second-unit bid.

4 Buyout option

If both buyers choose $q = 0$ in a buyout stage and Equation (1) holds, both buyers submit a single-unit bid in a bid stage as in Lemma 1. Thus, they each obtain one unit at zero price whenever the auction proceeds to a bid stage. However, a buyer has two major reasons to exercise a buyout option in a buyout stage. First, to exercise a buyout option is only the way for a buyer to obtain two units given a single-unit bid in a bid stage. Second, if an opponent tries to exercise a buyout option, a buyer may lose a chance to obtain even a single unit unless he also exercises an option, even though a single unit is enough for him. These reasons show the basic trade-off faced to a buyer between exercising a buyout option or not. In this section, we consider the optimal decision of buyer 1 on how many units to buy in a buyout stage. When buyer 1 exercises a buyout option and chooses $q > 0$, whether he can successfully buy the item(s) depends upon buyer 2's decision. With a little abuse of notation, we denote by $F(X_q)$ the probability that a buyer has valuations $(x_1, x_2) \in X_q$. We assume $F(X_2) + F(X_1) + F(X_0) = 1$.

First, suppose that buyer 1 chooses $q = 2$. There are three cases regarding buyer 2's choice. (i) If buyer 2 also chooses $q = 2$, which occurs with probability $F(X_2)$, buyer 1 then successfully obtains two units with probability 1/6, one unit with probability 4/6, and nothing with probability 1/6. Similarly, (ii) if buyer 2 chooses $q = 1$, which occurs with probability $F(X_1)$, buyer 1 successfully obtains two units with probability 1/3 and one unit with probability 2/3. On the other hand, (iii) if buyer 2 chooses $q = 0$, which occurs with probability $F(X_0)$, buyer 1 is awarded two units with certainty. Thus, given buyout price b, expected payoff $\Pi_2(x_1, x_2, b)$ of buyer 1 to choose $q = 2$ is given by

$$\begin{aligned}
\Pi_2(x_1, x_2, b) &= F(X_2)[\frac{1}{6}(x_1 + x_2 - 2b) + \frac{4}{6}(x_1 - b)] \\
&\quad + F(X_1)[\frac{1}{3}(x_1 + x_2 - 2b) + \frac{2}{3}(x_1 - b)] + F(X_0)(x_1 + x_2 - 2b) \\
&= [\frac{1}{6}F(X_2) + \frac{1}{3}F(X_1) + F(X_0)](x_1 + x_2 - 2b) + \frac{2}{3}[F(X_2) + F(X_1)](x_1 - b) \\
&= [1 - \frac{1}{6}F(X_2)](x_1 - b) + [\frac{1}{6}F(X_2) + \frac{1}{3}F(X_1) + F(X_0)](x_2 - b).
\end{aligned}$$

Note that the choice of $q = 2$ makes the game end with the buyout stage with certainty.

Second, suppose that buyer 1 chooses $q = 1$. There are two cases with regarding buyer 2's choice. (i) If buyer 2 chooses $q = 2$, buyer 1 is successfully awarded one unit with probability 2/3 whereas he obtains nothing with probability 1/3. On the other hand, (ii) if buyer 2 chooses $q < 2$, buyer 1 obtains one unit with certainty. Thus, expected payoff $\Pi_1(x_1, x_2, b)$ of buyer 1 to choose $q = 1$ is given by

$$\begin{aligned}
\Pi_1(x_1, x_2, b) &= F(X_2)\frac{2}{3}(x_1 - b) + [F(X_1) + F(X_0)](x_1 - b) \\
&= [1 - \frac{1}{3}F(X_2)](x_1 - b).
\end{aligned}$$

Third, suppose that buyer 1 does not exercise a buyout option and chooses $q = 0$. (i) If buyer 2 chooses $q = 2$, buyer 1 obtains nothing. On the other hand, (ii) if buyer 2 chooses $q < 2$, the auction proceeds to a bid stage and then buyer 1 obtains one unit at a zero price with certainty. Thus, expected payoff $\Pi_0(x_1, x_2, b)$ of buyer 1 choosing $q = 0$ is given by

$$\begin{aligned}
\Pi_0(x_1, x_2, b) &= [F(X_1) + F(X_0)](x_1 - 0) \\
&= [1 - F(X_2)]x_1.
\end{aligned}$$

Buyer 1 should optimally choose q in order to maximize his expected payoff derived above. We separately consider three cases regarding buyer 1's valuations: (i) $x_2 \leq x_1 < b$, (ii) $x_2 < b \leq x_1$, and (iii) $b < x_2 \leq x_1$. Buyer 1 never chooses $q > 0$ in the case (i) whereas he never chooses $q = 2$ in the case (ii). Note that $X_0 \neq \emptyset$ for any $b > 0$ because the case (i) realizes with a positive probability. The following proposition describes a symmetric undominated Bayesian equilibrium.

Proposition 1. Suppose that there is zero probability of ties that buyer 1's second-unit bid is equal to buyer 2's first-unit bid in a bid stage, and that Equation (1) is satisfied. Given buyout price b, there exists a symmetric undominated equilibrium in which a buyer with valuations $(x_1, x_2) \in X_q$ chooses quantity q in a buyout stage and submits single-unit bid $(p_1, p_2) = (x_1, 0)$ in a bid stage. The set of valuations X_q is given by

$$\begin{aligned} X_2 &= \{(x_1, x_2) \in X | x_2 \geq h(x_1), x_2 \geq b\}, \\ X_1 &= \{(x_1, x_2) \in X | x_1 \geq \underline{x}, x_2 \leq b\}, \\ X_0 &= X/(X_2 \cup X_1), \end{aligned}$$

where

$$h(x_1) = \frac{1}{2 - F(X_2) + 4F(X_0)}\Big(- 5F(X_2)x_1 + [8 - 2F(X_2) - F(X_0)]b\Big)$$

and

$$\underline{x} = \frac{3 - F(X_2)}{2F(X_2)}b.$$

Proof. Suppose that buyout price $b > 0$ is given. Without a loss of generality, we focus on a choice of buyer 1 with valuation (x_1, x_2). Since in the case (i) $x_2 \leq x_1 < b$ buyer 1 optimally never chooses $q > 0$ for any buyout price $b > 0$, we consider the cases (ii) $x_2 < b \leq x_1$ and (iii) $b < x_2 \leq x_1$.

In the case (ii), buyer 1 should choose $q < 2$ because the buyout price exceeds his second-unit valuation. Suppose that $X_2 \neq \emptyset$ holds for given $b > 0$. Let $\underline{x} \geq b$ denote buyer 1's first-unit valuation at which he is indifferent between choosing $q = 1$ and $q = 0$ in a buyout stage. The valuation $\underline{x}$ should satisfy

$$\begin{aligned} \Pi_1(\underline{x}, x_2, b) &= [1 - \frac{1}{3}F(X_2)](\underline{x} - b) \\ &= [1 - F(X_2)]\underline{x} = \Pi_0(\underline{x}, x_2, b). \end{aligned}$$

or equivalently

$$\underline{x} = \frac{3 - F(X_2)}{2F(X_2)}b.$$

Note that $\underline{x} > b$ holds for $F(X_2) \in (0, 1)$. Since the threshold $\underline{x}$ is uniquely determined by buyout price b, buyer 1 prefers $q = 1$ to $q = 0$ if and only if his first-unit valuation satisfies $x_1 > \underline{x}$. On the other hand, if $X_2 = \emptyset$ for the given buyout price $b > 0$, buyer 1 prefers $q = 0$ to $q = 1$ since $F(X_2) = 0$ yields $\Pi_1(x_1, x_2, b) = x_1 - b < \Pi_0(x_1, x_2, b) = x_1$ for any x_1.

In the case (iii), choosing $q = 1$ is strictly dominated by choosing $q = 2$, because $x_1 \geq x_2 > b$ yields

$$\Pi_2(x_1, x_2, b) - \Pi_1(x_1, x_2, b) = \frac{F(X_2)(x_1 - b)}{6} + [\frac{F(X_2)}{6} + \frac{F(X_1)}{3} + F(X_0)](x_2 - b) > 0.$$

Thus, let $(\hat{x}_1, \hat{x}_2)$ denote the valuations of buyer 1 at which he is indifferent between choosing $q = 2$ and $q = 0$. The valuations should satisfy

$$[1 - \frac{1}{6}F(X_2)](\hat{x}_1 - b) + [\frac{1}{6}F(X_2) + \frac{1}{3}F(X_1) + F(X_0)](\hat{x}_2 - b) = [1 - F(X_2)]\hat{x}_1,$$

or equivalently

$$\hat{x}_2 \equiv h(\hat{x}_1) = \frac{1}{2 - F(X_2) + 4F(X_0)}\Big(- 5F(X_2)\hat{x}_1 + [8 - 2F(X_2) - F(X_0)]b\Big). \tag{2}$$

The set of $(\hat{x}_1, \hat{x}_2)$ satisfying Equation (2) is represented by a downward-sloping linear line in the (x_1, x_2)-plane. By a simple calculation, we obtain $h(b) = ([8 - 7F(X_2) - F(X_0)]/[2 - F(X_2) + 4F(X_0)])b > b$ and $h(\underline{x}) = ([1 + F(X_2) - 2F(X_0)]/[2(2 - F(X_2) + 4F(X_0))])b < b$. Therefore, $x_2 = h(x_1)$ intersects $x_2 = b$ at $x_1 = x^* \in (b, \underline{x})$. Specifically, $x^* = b[6 - F(X_2) - 5F(X_0)]/(5F(X_2))$.

Since these thresholds $\underline{x}$ and $\hat{x}_2 = h(\hat{x}_1)$ are uniquely determined by buyout price b and yield the triplet of sets (X_2, X_1, X_0). Thus, by construction, buyer 1 has no incentives to deviate from choosing q based on (X_2, X_1, X_0). In a bid stage, as Lemma 1 shows, both buyers cannot profitably deviate from submitting single-unit bid as long as Equation (1) is satisfied. □

Note that even if a second-unit valuation exceeds a buyout price, a buyer may not choose $q = 2$ or $q = 1$. This result is analogized to the results obtained in single-unit auctions. Figure 1 illustrates the region of valuations that a buyer chooses q in the symmetric undominated Bayesian equilibrium. Equation (2) is depicted by a downward-sloping line in Figure 1. The $x_2 = h(x_1)$ intersects $x_2 = x_1$ at the point A, which is above $x_2 = b$ because of $h(b) > b$. The point B is given by (x^*, b). As illustrated in the figure, a buyer with higher valuations tend to exercise a buyout option for more units.

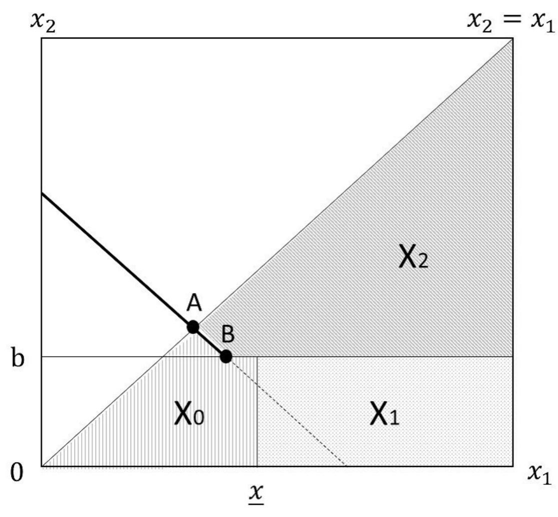

Figure 1: Thresholds in case of $X_2 \neq \emptyset$

Unfortunately, it is ambiguous whether X_2 shrinks as b increases because X_q, h, and $\underline{x}$ affect each other. However, we provide a necessary condition that a buyout option is never exercised. First, we show that if a buyer chooses $q < 2$ for some buyout price b in equilibrium, then a buyout option is never exercised for such a buyout price.

Proposition 2. Suppose that there exists a symmetric undominated Bayesian equilibrium presented in Proposition 1. In the equilibrium, if $X_2 = \emptyset$, then $X_0 = X$.

Proof. Suppose that in the symmetric undominated Bayesian equilibrium presented in Proposition 1, $X_2 = \emptyset$ for some $b > 0$. We separately consider three cases regarding buyer 1's valuations and show that he never chooses $q = 1$ for such a buyout price. By Proposition 1, $X_2 = \emptyset$ implies either $x_2 < b$ or $x_2 < h(x_1)$ given b. First, if buyer 1 has valuations satisfying (i) $x_2 \leq x_1 < b$, he never chooses $q > 0$. Second, if his valuations satisfy (ii) $x_2 < b \leq x_1$, he prefers $q = 0$ to $q = 1$ because $F(X_2) = 0$ leads to $\Pi_0(x_1, x_2, b) = x_1 > \Pi_1(x_1, x_2, b) = x_1 - b$. Finally, if his valuations satisfy (iii) $b < x_2 \leq x_1$, choosing $q = 2$ dominates $q = 1$ because $\Pi_2(x_1, x_2, b) > \Pi_1(x_1, x_2, b)$ for any $b < x_2 \leq x_1$, as in the proof of Proposition 1. Moreover, $x_2 < h(x_1)$ implies that a buyer prefers $q = 0$ to $q = 2$. Thus, buyer 1 never chooses $q = 1$. Since the equilibrium is symmetric, $X_1 = \emptyset$ should hold. Hence $X_0 = X$. $\square$

Proposition 2 says that if a buyout price is sufficiently high that a buyer never chooses $q = 2$, a buyout option is never exercised for such a buyout price. Note that a first-unit valuation or both units valuations for a buyer may exceed such a buyout price. However, Proposition 2 states that a buyer never exercises a buyout option. The underlying intuition is clear. A buyer is ensured one unit at a zero price as long as his opponent never chooses $q = 2$. Thus, he has no incentive to buy one unit at a (positive) buyout price, resulting in a choice of $q = 0$.

Note that although $X_2 = \emptyset$ implies $X_1 = \emptyset$, the converse is not true. Thus, if $X_1 = \emptyset$,, there exist three cases: (1) a buyout option is never exercised, (2) a buyer chooses $q \in \{0, 2\}$, and (3) a buyer chooses $q \in \{0, 1, 2\}$. The next proposition gives a necessary condition that a buyout option is never exercised.

Proposition 3. Suppose that there exists a symmetric undominated Bayesian equilibrium presented in the proposition 1. In the equilibrium, if $X_0 = X$ (i.e., $F(X_0) = 1$), then $b > \frac{1}{2}\bar{x}$.

Proof. Suppose that in the symmetric undominated Bayesian equilibrium, $X_0 = X$ for some $b > 0$. Since $F(X_2) = F(X_1) = 0$, buyer 1's expected payoffs are rewritten as $\Pi_2(x_1, x_2, b) = x_1 + x_2 - 2b$

and $\Pi_0(x_1, x_2, b) = x_1$. Note that $q = 1$ is strictly dominated by $q = 2$ for a buyer with valuations that satisfies $b < x_2 \leq x_1$. Buyer 1 should choose $q = 0$ if and only if $\Pi_0(x_1, x_2, b) > \Pi_2(x_1, x_2, b)$ for any (x_1, x_2). That is, $b > \frac{1}{2}\bar{x}$. $\square$

Proposition 3 implies that a second-unit valuation plays a crucial role for a buyer to decide whether to exercise a buyout option. Since seller revenue is zero without a buyout price, we immediately see the revenue-enhancing effect of a buyout price from the proposition 3.

Corollary 1. Any buyout price $b \leq \frac{1}{2}\bar{x}$ is exercised with a positive probability and increases the expected seller revenue.

Unfortunately, we cannot explicitly calculate the degree of revenue-enhancing effect even in our simple model. Thus, we consider a special case with flat-demand to observe the revenue-enhancing effect in the next section.

5 Flat-demand example

In this section, we consider a special case where a buyer has flat-demand (i.e., $x_1 = x_2 = x$) and his valuation is uniformly distributed on $[0, 1]$. As noted in Section 3, assumptions of a flat-demand and a uniform distribution satisfy Equation (1) in Lemma 1. Specifically, suppose that a buyer with valuation x submits $p \in [0, 1]$ for a second unit. Since $F_1(x) = F(x) = x$ and $f_1(x) = 1$, we have

$$(x - p)\frac{f_1(p)}{1 - F_1(p)} = \frac{x - p}{1 - p} \leq 1$$

for any x and for any $p \leq x$. For the next step, we define $\bar{X} = [0, \bar{x}]$ for any $\bar{x} \in [0, 1]$. Since $F_1(x|\bar{x}) = \frac{x}{\bar{x}}$ and $f_1(x|\bar{x}) = \frac{1}{\bar{x}}$, we have

$$(x - p)\frac{f_1(p|\bar{x})}{1 - F_1(p|\bar{x})} = \frac{x - p}{\bar{x} - p} \leq 1$$

for any $x \in \bar{X} = [0, \bar{x}]$ and for any $p \leq x$. Therefore, by Lemma 1, single-bid $(x, 0)$ in a bid stage constitutes a symmetric undominated Bayesian equilibrium.

Since Lemma 1 applies to this case, we should consider two thresholds $\bar{c}$ and $\underline{c}$. Given buyout price b, a buyer chooses $q = 2$ if his valuation satisfies $x \in [\bar{c}, 1]$, $q = 1$ if $x \in [\underline{c}, \bar{c})$, and $q = 0$ if $x \in [0, \underline{c})$ in a bid stage. Since a buyer never chooses $q > 0$ if $x < b$, $\underline{c} \geq b$ should hold. We consider an equilibrium consisted of these thresholds $\bar{c}$ and $\underline{c}$ with $\bar{c} \geq \underline{c}$.

The expected payoffs of a buyer to choose $1 \in \{0, 1, 2\}$ are given by

$$\begin{aligned}
\Pi_2(x,b) &= (1-\bar{c})[\frac{1}{6}(2x-2b)+\frac{4}{6}(x-b)] + (\bar{c}-\underline{c})[\frac{1}{3}(2x-2b)+\frac{2}{3}(x-b)] + \underline{c}(2x-2b) \\
&= (1+\frac{1}{3}\bar{c}+\frac{2}{3}\underline{c})(x-b), \\
\Pi_1(x,b) &= (1-\bar{c})\frac{2}{3}(x-b) + \bar{c}(x-b) \\
&= (\frac{2}{3}+\frac{1}{3}\bar{c})(x-b), \\
\Pi_0(x,b) &= \bar{c}x.
\end{aligned}$$

For a buyer with valuation $x \geq b$, choosing $q = 1$ is strictly dominated by choosing $q = 2$ since $\Pi_2(x, b) > \Pi_1(x, b)$; thus, we suffice to compare two choices $q = 2$ and $q = 0$. Let c denote a threshold that a buyer chooses $q = 2$ if his valuation satisfies $x \geq c$. The expected payoffs can be rewritten as

$$\begin{aligned}
\Pi_2(x,b) &= (1-c)[\frac{1}{6}(2x-2b)+\frac{4}{6}(x-b)] + c(2x-2b) = (1+c)(x-b), \\
\Pi_0(x,b) &= cx.
\end{aligned}$$

Since a buyer should be indifferent between choosing $q = 2$ and $q = 0$ if his valuation is c, we obtain

$$(1+c)(c-b) = c^2,$$

or equivalently $c = b/(1-b)$. Figure 2 illustrates the relation between buyout price b and threshold c. The horizontal and vertical axises represent b and c, respectively. As shown in Figure 2, threshold c increases with buyout price b, and a buyout price is never exercised if $b > 1/2$ because $c > 1$ holds for $b > 1/2$. The seller revenue corresponding buyout price b is given by

$$U(b) = \Big[1 - (\frac{b}{1-b})^2\Big]2b = \frac{2(1-2b)b}{(1-b)^2}.$$

Thus, the first-order condition yields the optimal buyout price $b^* = 1/3$, leading to $U(1/3) = 1/2$. Figure 3 illustrates the expected seller revenue. In Figure 3, the horizontal and vertical axises represent b and U, respectively. The dotted line represents $U(b)$ in Figure 3. The buyout option weakly improves seller revenue since the single-unit bid yields a zero revenue. Note that a sincere bid on both units yield a higher seller revenue (and perfect efficiency) which is depicted by a solid line in the figure. If buyers are forced to submit a flat-bid (i.e., $(p_1, p_2) = (p, p)$), a dominant strategy for them is to bid sincerely, results in improvement of seller revenue. To exercise a buyout option has a perspective of a flat-bid; thus, seller revenue increases with the introduction of a buyout option.

6 Concluding remarks

This paper studies the impact of a buyout price on strategies of buyers and seller revenue in uniform-price sealed-bid auctions and suggest that the introduction of a buyout price improves seller revenue by

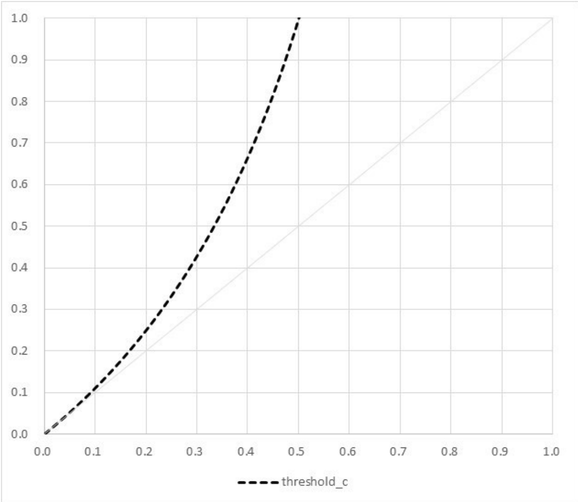

Figure 2: Threshold

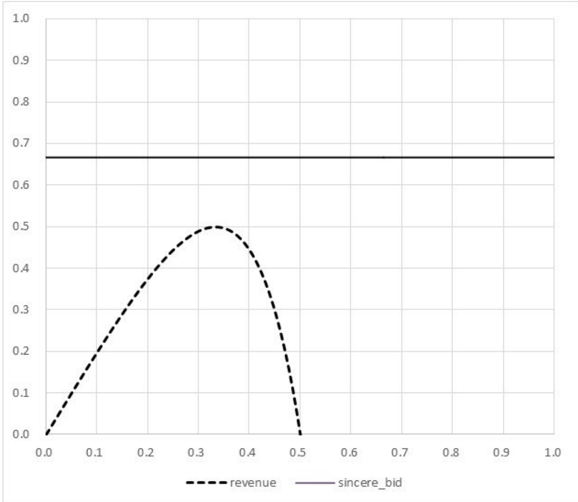

Figure 3: Expected seller revenue

deterring a demand reduction. As well known, seller revenue tends to be low in uniform-price auctions because buyers lower a bid on a second unit. This phenomenon is known as a demand reduction, and as the extreme case, theoretically, buyers submit zero price for a second unit, yielding zero revenue to a seller. A demand reduction is prevalent in the experimental and the real world multi-unit auctions, and thus the low revenue caused by a demand reduction matters. This paper suggests that a buyout price solves this issue. Multi-unit auctions have grown increasingly important in terms of a practical usage, and thus the result obtained in this paper is beneficial to auctioneers and auction designers.

A buyout option can perform better in multi-unit auctions than in single-unit ones. Chen et al. (2013) study a buyout price in second-price sealed-bid auctions where a risk-averse buyer has single-unit demand and show that a buyout price improves seller revenue. They discuss that a seller can be seriously harmed by posting an inappropriate buyout price. In the reality, such a mis-pricing can happen if a seller does not have enough information about the distribution of valuation or the degree of risk aversion. However, this does not matter in multi-unit auctions when low revenues indeed realize. In this sense, the introduction of a buyout price is less “risky” in multi-unit uniform-price auctions as compared with single-unit auctions.

We consider a simple situation in this paper, and thus the analysis has a clear limitation. First, we restrict our attention to the case where buyers bid zero on a second unit. However, the revenue-enhancing effect of buyout prices do not necessarily rely on a single-unit bid. A buyout price creates a trade-off faced to buyers as long as a demand reduction emerges, although the revenue-enhancing effect is moderated.

Second, we focus on the situation where only two buyers are involved. It seems that a buyout price can improve seller revenue as long as a demand reduction occurs even though more buyers participate auctions. A field experiment conducted by Engelbrecht-Wiggans et al. (2006) seems to support this conjecture. In their field experiment, three or five bidders with two-unit demand compete in bids for sports trading cards in uniform-price and Vickrey sealed-bid auctions. They compare the results with those of List and Lucking-Reiley (2000) who consider a two-bidder case. Their major finding is that in most cases, bidders on average submit lower second-unit bids in the uniform-price auctions than in the Vickrey auctions. Although a change of the number of bidders from three to five increases the second-unit bid in the uniform-price sealed-bid auctions, they report that these differences are statistically insignificant.

Third, we focus on buyers demanding at most two units of items. As List and Lucking-Reiley (2000) note that an increase in the number of items intensifies a demand reduction. Thus, a buyout price might play a more effective role to prevent seller revenue from suffering from a demand reduction. Clearly, a two-buyer two-item case is just a first step, and a sophisticated model is required for overcoming these three limitations. However, analyzing a general case yields the difficulty to arise at considering a buyout stage because calculating a winning probability to obtain a certain unit becomes complicated.

Apart from these topics, more work remains for the future research. Since our model is simple, we can conduct an experiment and test the revenue-enhancing effect of a buyout price in uniform-price sealed-bid auctions. The experimental design of single-unit auctions with a buyout price (e.g., Ivanova-Stenzel and Kroger, 2008) can directly apply to multi-unit auctions with a buyout price. As for multi-unit auctions in the actual world, it seems important to consider the impact of a buyout price on seller revenue in multi-unit ascending auctions. In fact, the experimental studies report a more considerable level of demand reductions (and hence low seller revenue) in ascending auction (Alsemgeest et al., 1998; Porter and Vragov, 2006; Engelmann and Grimm, 2009). Moreover, the FCC spectrum and the second-generation spectrum auctions that present a demand reduction are multi-unit ascending auctions. We await the further understanding of multi-unit auctions with a buyout option from future researches.

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